NAME:

CS 336, Exam 1, February 5, 2002

This exam counts for 20% of your course grade. You must do this exam without notes, calculators, and without consulting any other source than your own intellect (so don't look at exam papers other than your own!).

SHOW ALL YOUR WORK as we will, in some cases, give partial credit.

In questions 1 through 4 you will be asked to define various terms. For these questions, the permitted symbols you will use are these:

- numbers
- letters (lowercase and uppercase, in English and in Greek) to represent objects (e.g. sets or elements of sets)
- Logical symbols, such as the symbols for and, or, not, implies, if-and-only-if (and anything else I'm forgetting that's standard).
- Standard notations for sets, including $\{, \}, s.t., \in, \subseteq,$, $\cup$, $\cap$, $\emptyset$, and anything else that is standard (in case I forgot something).
- $|A|$, i.e. the cardinality of A, meaning the number of elements of A
- $A \times B$, for the cross product of A and B (and its extension to an k-way cross product).
- (and), as in (a, b, c), as an element of $A \times B \times C$.
- $\mathcal{P}(S)$, the "power set" of S (i.e. the set of all subsets of S)
- $\emptyset$, the "empty set".
- Inequalities, so $>, \geq, \leq, =$.
- Binary arithmetic operations: $a \times b, a \div b, a + b, a - b$.
- quantifiers: $\forall$ and $\exists$.

Please ask me explicitly if there is some other symbol you'd like to use that I haven't listed.

You will have 60 minutes for the exam, and then the exams will be collected.

1. (5 points) Define the set $\mathcal{P}$ of primes using the permitted notation.

2. (6 points) Using the definition of the set $\mathcal{P}$ from problem 1, define the set of numbers that are the product of two distinct primes, using the permitted notation.

3. (7 points) Recall the representation of a function f as a binary relation R_f, whereby $< a, b > \in R_f$ if and only if $f(a) = b$. *Using this representation* (i.e. by referring only to R_f), define what it means for a function mapping a set A to itself to be one-to-one, using the permitted notation.

4. (20 points) Consider the following situation. You are given a group of people and you know, for every pair of people, whether they are friends or not (assume friendship is symmetric). You are given a target number, k, and you want to find a group of k people in the class that have the maximum total number of friends. Describe this as a graph-theoretic question. What are the vertices, what are the edges, and what are you trying to solve?

5. (6 points) Let $S = \{1, 2, 3\}$. Let $A = \{x \subseteq S : |x| = 1 \Rightarrow |x| = 2\}$. List all elements of A.

6. Define the binary relation X on the subsets of $\{1, 2, 3\}$ to contain those ordered pairs $< a, b >$ where a and b are not disjoint. Answer the following questions about X:

   (a) (8 points) List all the elements of X.

   (b) (8 points) Let the set C be defined by:

   $$C = \{x \subseteq \{1, 2, 3\} : \nexists y \ s.t. \ < x, y > \in X \ and \ x \neq y\}.$$

   List the elements of C.

7. (20 points) Prove by contradiction ONLY ONE of the following two assertions: (a) there are an infinite number of primes OR (b) $\sqrt{2}$ is irrational.

8. (20 points) Prove by induction that for all non-negative integers n, $n^3 - n$ is divisible by 3.