CS 336, Questions about Graphs

1. Consider the following definition of a company being "happy enough": *Everyone in each company likes at least as many people in their company as they dislike (not counting themselves). By contrast, an "unhappy" company has at least one person who dislikes more people in the company than they like (not counting themselves).*

   Note the following assumptions:

   - "Like" is symmetric (i.e. if person A likes person B, then person B likes person A)
   - Each person neither likes nor dislikes himself or herself. Thus, if there are two people in the company and they dislike each other then the company is unhappy (because we consider each person to neither like nor dislike himself or herself, each dislikes more people in the company than they like).
   - For every two people in the company, either they each like or each dislike the other (no two people are indifferent to each other).

   This question has several parts:

   (a) Is a company of one person a happy enough company, or an unhappy company?

   (b) Describe a company and its interpersonal relationships as a graph (what are the vertices, and what are the edges), and then state the "happy enough" condition graph-theoretically (i.e. it is a property that can hold or not hold for any given graph).

   (c) Give an example (as a graph) of an unhappy company which can be divided into two happy enough companies.

   (d) Give an example (as a graph) of an unhappy company which cannot be divided into two happy enough companies.

2. A **proper vertex coloring** of a graph $G = (V, E)$ is an assignment of integers (or colors) to the nodes of the graph G so that every pair of adjacent nodes are assigned different colors.

   - If we define a k-coloring of the graph $G = (V, E)$ as a function $f : V \to \{1, 2, \ldots, k\}$, then state the condition (mathematically) on f which makes it a "proper vertex coloring."
   - Draw a graph which has a proper 1-coloring. What condition must all graphs satisfy to have a proper 1-coloring?
   - Draw a graph which has a proper 2-coloring but no proper 1-coloring.
   - Draw a graph which has a proper 3-coloring but no proper 2-coloring.

3. The **chromatic number** of a graph G is the minimum number of colors which can be used to properly vertex color G. Define the chromatic number of G mathematically (i.e. formally).

4. Draw each of the following graphs:

   (a) $G = (V, E) : V = \{v_1, v_2, \ldots, v_{10}\} E = \{(v_i, v_j) : i + j = 11\}$

   (b) $G = (V, E) : V = \{v_1, v_2, \ldots, v_{10}\} E = \{(v_i, v_j) : ij = 10\}$

   (c) $G = (V, E) : V = \{v_1, v_2, \ldots, v_{10}\} E = \{(v_i, v_j) : i \textit{ divides } j\}$

   (d) $G = (V, E) : V = \{v_1, v_2, \ldots, v_{10}\} E = \{(v_i, v_j) : gcd(i, j) \neq 1\}$

   (e) $G = (V, E) : V = \{v_1, v_2, \ldots, v_{10}\} E = \{(v_i, v_j) : gcd(i, j) = 1\}$

5. For each of the graphs in the previous problem, compute the number of components, the degree sequence, and the chromatic number.

6. You are the manager of a large team of 100 computer programmers, and you need to divide this large team into a collection of smaller teams so that no two people on the same team are enemies, but you'd like to divide the large team into as few teams as possible. Let us assume that you know who the enemies are. Describe this problem graph-theoretically. What are the vertices, what are the edges, and what are you looking for?

7. You are the village matchmaker, and you need to marry people off to earn your daily living. For every marriage you create you will earn $1000. All the unmarried people in the village indicate who they are willing to marry, and they have promised not to complain as long as you select someone from their list; however, if you select someone not on their list of acceptable mates, the marriage will not be performed. You can only marry girls to boys (no same sex marriages in this village), and you can only create monogamous marriages. There are no divorces in this village. Describe this problem graph-theoretically. What are the vertices, what are the edges, and what are you looking for?

8. You are a travelling salesman and you need to travel to every city in a collection of cities. You know the number of miles between every pair of cities, and you want to minimize the total number of miles you travel. Describe this problem graph-theoretically. What are the vertices, what are the edges, and what are you looking for?

9. For each of the graphs $G = (V, E)$ at the bottom of the page, and for each of the following properties, identify a smallest subset $A \subseteq V$ which satisfies the property:

   (a) $\forall x \in V - A \; \exists y \in A \; s.t. \; (x, y) \in E$

   (b) $\forall (x, y) \in E \; \exists a \in A \; s.t. \; (x, a) \in E \; or \; (y, a) \in E$

10. Describe each of the following "real-world" problems graph theoretically.

    (a) You are the boss of a large company. You want to divide your company into a small number of companies. You want everyone in each small company to get along, but you want to minimize the number of small companies you form. Assume you know for every pair of persons whether they get along. What graph theoretic problem do you need to solve?

    (b) You want to spread a rumor, and you know that if you tell a person, they will tell everyone they know the rumor, but that each person will only repeat the rumor if they are told it by you, personally. You know who knows whom in your class. What is the minimum number of people you have to tell, in order for everyone to be told the rumor?