CS 336, Homework 1

This homework covers pre-requisites for the course.

You will be asked to define various terms. The permitted symbols you will use are these:

- numbers
- letters (lowercase and uppercase, in English and in Greek) to represent objects (e.g. sets or elements of sets)
- Logical symbols, such as the symbols for and, or, not, implies, if-and-only-if (and anything else I'm forgetting that's standard).
- Standard notations for sets, including $\{, \}, s.t., \in, \subseteq,$, $\cup$, $\cap$, $\emptyset$, and anything else that is standard (in case I forgot something).
- $|A|$, i.e. the cardinality of A, meaning the number of elements of A
- $A \times B$, for the cross product of A and B (and its extension to an k-way cross product).
- (and), as in (a, b, c), as an element of $A \times B \times C$.
- $\mathcal{P}(S)$, the "power set" of S (i.e. the set of all subsets of S)
- $\emptyset$, the "empty set".
- Inequalities, so $>, \geq, \leq, =$.
- Arithmetic operations $+, -$, etc.
- quantifiers: $\forall$ and $\exists$.
- Predictates Px for P a property (expressed logically).

Please ask me explicitly if there is some other symbol you'd like to use that I haven't listed.

You will have 40 minutes for the exam, and then the exams will be collected.

1. Let $A = \{\{0, 1\}, \{1\}, \{2, 3\}, \{\}, \{2\}\}$. Let R be a relation on A defined as $R = \{(x, y) | x, y \in A, |x| = |y|\}$.

   (a) List all elements of R.

   (b) What properties does R have ?

2. Let $S = \{1, 2, 3\}$. Let $A = \{x \subseteq S : 1 \in x \Rightarrow 1 \notin x\}$. List all elements of A.

3. Let $S = \{1, 2, 3, 4, 5\}$. Let $A = \{x \in S : \exists y \in S, x^2 < y\}$. List elements of A.

4. Let $S = \{1, 2, 3, 4, 5\}$. Let $A = \{x \in S : \forall y \in S, x^2 < y\}$. List all elements of A.

5. Let $S = \{1, 2, 3, 4, 5\}$. Let $A = \{x \in S : \exists y \in S, \exists z \in S, y < z\}$. List all elements of A.

6. Using the permitted symbols, express the set of all prime numbers.

7. Given a function $f : A \to B$, express the set of elements in B which are mapped to by f from at least two elements in A.

8. Let A be a set of people. Let R be the relation on A for "friendship". Define the set of friends of x (for $x \in A$) to be those elements y in A so that there is some pair in R in which x is paired with y. Define this set using formal notation.

9. Let A be a set of people. Let R be the relation on A for "friendship". Define the "friendliest person" to be that person in A with the most friends. Define this formally in terms of R.

10. Let A be a set of people. Let R be the relation on A for "friendship". We will say that A is a "friendly bunch of people" if everyone in A is friends with at least as many people as they are not friends with. Define this formally.

11. Let Z denote the set of integers. Express the set of all even integers.

12. Prove that for all sets A, B, C, and D,

$$(A - B) - C \subseteq A - (B \cup C).$$

13. Let $r \neq 1$. Prove that for all $n \in N$, $\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$.