

CS 336, Homework 1 - solutions

1. Let  $A = \{\{0, 1\}, \{1\}, \{2, 3\}, \{\}, \{2\}\}$ . Let  $R$  be a relation on  $A$  defined as  $R = \{(x, y) | x, y \in A, |x| = |y|\}$ .

(a) List all elements of  $R$ .

$$R = \{(\{\}, \{\}), (\{1\}, \{1\}), (\{2\}, \{2\}), (\{0, 1\}, \{0, 1\}), (\{2, 3\}, \{2, 3\}), (\{1\}, \{2\}), (\{0, 1\}, \{2, 3\})\}$$

(b) What properties does  $R$  have ?

reflexive, transitive, symmetric

2. Let  $S = \{1, 2, 3\}$ . Let  $A = \{x \subseteq S : 1 \in x \Rightarrow 1 \notin x\}$ . List all elements of  $A$ .

$$A = \{\{\}, \{2\}, \{3\}, \{2, 3\}\}$$

3. Let  $S = \{1, 2, 3, 4, 5\}$ . Let  $A = \{x \in S : \exists y \in S, x^2 < y\}$ . List elements of  $A$ .

$$A = \{1, 2\}$$

4. Let  $S = \{1, 2, 3, 4, 5\}$ . Let  $A = \{x \in S : \forall y \in S, x^2 < y\}$ . List all elements of  $A$ .

$$A = \{\}$$

5. Let  $S = \{1, 2, 3, 4, 5\}$ . Let  $A = \{x \in S : \exists y \in S, \exists z \in S, y < z\}$ . List all elements of  $A$ .

$$A = \{1, 2, 3, 4, 5\}$$

6. Using the permitted symbols, express the set of all prime numbers.

$$\{n \in \mathcal{N} : \forall x \in \mathcal{N}, ((x > 1 \wedge x < n) \Rightarrow x \nmid n)\}$$

or

$$\{n \in \mathcal{N} : \neg \exists x, y \in \mathcal{N}, (x > 1 \wedge x \leq y \wedge y < n \wedge xy = n)\}$$

7. Given a function  $f : A \rightarrow B$ , express the set of elements in  $B$  which are mapped to by  $f$  from at least two elements in  $A$ .

$$\{b \in B : \exists \{a_1, a_2\} \subseteq A, (f(a_1) = b \wedge f(a_2) = b)\}$$

8. Let  $A$  be a set of people. Let  $R$  be the relation on  $A$  for “friendship”. Define the set of friends of  $x$  (for  $x \in A$ ) to be those elements  $y$  in  $A$  so that there is some pair in  $R$  in which  $x$  is paired with  $y$ . Define this set using formal notation.

$$\text{friends} : A \rightarrow \mathcal{P}(A)$$

$$\text{friends}(x) = \{y \in A : (x, y) \in R\}$$

9. Let  $A$  be a set of people. Let  $R$  be the relation on  $A$  for “friendship”. Define the “friendliest person” to be that person in  $A$  with the most friends. Define this formally in terms of  $R$ .

$$x \in A : \forall y \in A, |\text{friends}(x)| \geq |\text{friends}(y)|$$

10. Let  $A$  be a set of people. Let  $R$  be the relation on  $A$  for “friendship”. We will say that  $A$  is a “friendly bunch of people” if everyone in  $A$  is friends with at least as many people as they are not friends with. Define this formally.

$$\forall x \in A, |\text{friends}(x)| \geq |A| - |\text{friends}(x)| - 1$$

11. Let  $Z$  denote the set of integers. Express the set of all even integers.

$$\{n \in \mathcal{Z} : \exists x \in \mathcal{Z}, 2x = n\}$$

12. Prove that for all sets  $A, B, C$ , and  $D$ ,

$$(A - B) - C \subseteq A - (B \cup C).$$

For any  $x$ ,

$x \in (A - B) - C$  iff

$x \in A - B \wedge x \notin C$  iff

$x \in A \wedge x \notin B \wedge x \notin C$  iff

$x \in A \wedge x \notin B \cup C$  iff

$x \in A - (B \cup C)$

So  $(A - B) - C = A - (B \cup C)$ , and then  $(A - B) - C \subseteq A - (B \cup C)$ .

13. Let  $r \neq 1$ . Prove that for all  $n \in \mathbb{N}$ ,  $\sum_{i=0}^n r^i = \frac{r^{n+1}-1}{r-1}$ .

Proof by mathematical induction on  $n$ .

- base case  $n = 0$ .

$$\sum_{i=0}^0 r^i = r^0 = 1,$$

$$\frac{r^{0+1}-1}{r-1} = \frac{r-1}{r-1} = 1.$$

- for  $n + 1$ , assuming by induction hypothesis that the assertion holds for  $n$ .

$$\sum_{i=0}^{n+1} r^i = r^{n+1} + \sum_{i=0}^n r^i = r^{n+1} + \frac{r^{n+1}-1}{r-1} = \frac{r^{n+1}(r-1)+r^{n+1}-1}{r-1} = \frac{r^{n+2}-r^{n+1}+r^{n+1}-1}{r-1} = \frac{r^{n+2}-1}{r-1}.$$

14. (additional exercise) Let  $R = \{ \langle f, g \rangle : f : \mathcal{R} \rightarrow \mathcal{R} \wedge g : \mathcal{R} \rightarrow \mathcal{R} \wedge \exists \{a, b\} \subset \mathcal{R}, (f(a) = g(b) \wedge f(b) = g(a)) \}$ . Prove or disprove that  $\forall f : \mathcal{R} \rightarrow \mathcal{R} \exists g : \mathcal{R} \rightarrow \mathcal{R}, \langle f, g \rangle \in R$

Proof:

For any  $f$ , take  $g(x) = f(-x)$  for any  $x$ . Then choose an  $a \neq 0$  and  $b = -a$ . We have that  $g(b) = f(-b) = f(a)$ . Analogously  $g(a) = f(-a) = f(b)$ . Consequently  $\langle f, g \rangle \in R$ .

The assertion of the problem is a direct consequence of that.