Strategyproof Pareto-Stable Mechanisms for Two-Sided Matching with Indifferences*

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Abstract

We study variants of the stable marriage and college admissions models in which the agents are allowed to express weak preferences over the set of agents on the other side of the market and the option of remaining unmatched. For the problems that we address, previous authors have presented polynomial-time algorithms for computing a "Pareto-stable" matching. In the case of college admissions, these algorithms require the preferences of the colleges over groups of students to satisfy a technical condition related to responsiveness. We design new polynomial-time Pareto-stable algorithms for stable marriage and college admissions that correspond to strategyproof mechanisms. For stable marriage, it is known that no Pareto-stable mechanism is strategyproof for all of the agents; our algorithm provides a mechanism that is strategyproof for the agents on one side of the market. For college admissions, it is known that no Pareto-stable mechanism can be strategyproof for the colleges; our algorithm provides a mechanism that is strategyproof for the students.

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1 Introduction

Gale and Shapley [4] introduced the stable marriage model and its generalization to the college admissions model. Their work spawned a vast literature on two-sided matching; see Manlove [6] for a recent survey. The present paper is primarily concerned with variants of the stable marriage and college admissions models where the agents have weak preferences, i.e., where indifferences are allowed.

In the most basic stable marriage model, we are given an equal number of men and women, where each man (resp., woman) has complete, strict preferences over the set of women (resp., men); we refer to this model as SMCS. For SMCS, an outcome is a matching that pairs up all of the men and women into disjoint man-woman pairs. A man-woman pair (p, q) is said to form a *blocking pair* for a matching M if p prefers q to his partner in M and q prefers p to her partner in M. A matching is *stable* if it does not have a blocking pair. It is straightforward to prove that any stable matching is also Pareto optimal. Gale and Shapley presented the deferred acceptance (DA) algorithm for the SMCS problem and proved that man-proposing version of the DA algorithm produces the unique man-optimal (and woman-pessimal) stable matching. Roth [7] showed that the associated mechanism, which we refer to as the *man-proposing DA mechanism*, is *strategyproof* for the men, i.e., it is a weakly dominant strategy for each man to declare his true preferences. Unfortunately, the man-proposing DA mechanism is not strategyproof for the women. In fact, Roth [7] showed that no stable mechanism for SMCS is strategyproof for all of the agents.

The SMCW model is the generalization of the SMCS model in which each man (resp., woman) has weak preferences over the set of women (resp., men). When indifferences are allowed, we need to refine our notion of a blocking pair. A man-woman pair (p, q) is said to form a *strongly blocking pair* for a matching M if p prefers q to his partner in M and q prefers p to her partner in M. A matching is *weakly stable* if it is individually rational as defined in Sect. 4 and it does not have a strongly blocking pair. Two other natural notions of stability, namely strong stability and super-stability, have been investigated in the literature (see Manlove [6, Chapter 3] for a survey of these results). We focus on weak stability because every SMCW instance admits a weakly stable matching (this follows from the existence of stable matchings for SMCS, coupled with arbitrary tie-breaking), but not every SMCW instance admits a strongly stable or super-stable matching. It is straightforward to exhibit SMCW instances (with as few as two men and two women) for which some weakly stable matching is not Pareto optimal. Sotomayor [11] proves that every SMCW instance admits a matching that is Pareto optimal, and argues that *Pareto-stability* (i.e., Pareto optimality plus weak stability) is an appropriate solution concept for SMCW and certain other matching models with weak preferences.

Erdil and Ergin [2] and Chen and Ghosh [1] present polynomial-time algorithms for computing a Pareto-stable matching of a given SMCW instance; in fact, these algorithms are applicable to certain more general models to be discussed shortly. Given the existence of a stable mechanism for SMCS that is strategyproof for the men (or, symmetrically, for the women), it is natural to ask whether there is a Pareto-stable mechanism for SMCW that is strategyproof for the men. We cannot hope to find a Pareto-stable mechanism for SMCW that is strategyproof for all agents, since that would imply a stable mechanism for SMCS that is strategyproof for all agents. A similar statement holds for the SMIW model, the generalization of the SMCW model in which the agents are allowed to express incomplete preferences. See Sect. 4 for a formal definition of the SMIW model and the associated notions of weak stability and Pareto-stability. Throughout the remainder of the paper, when we say that a mechanism for a stable marriage model is strategyproof, we mean that it is strategyproof for the agents on one side of the market; moreover, unless otherwise specified, it is to be understood that the mechanism is strategyproof for the men. This paper provides the first Pareto-stable mechanism for SMIW (and also SMCW) that is shown to be strategyproof. Our SMIW mechanism generalizes Gale and Shapley's DA mechanism, and admits a polynomial time implementation.

The college admissions model with weak preferences, which we denote CAW, is a further generalization of the SMIW model. In the CAW model, students and colleges are being matched rather than men and women, and each college has a positive integer capacity representing the number of students that it can accommodate. See Sect. 5 for a formal definition of the CAW model and the associated notions of weak stability and Pareto-stability.

A key difference between CAW and SMIW is that in addition to expressing preferences over individual students, the colleges have preferences over *groups* of students. This characteristic is shared by the CAS model, which is the restriction of the CAW model to strict preferences. It is known that no stable mechanism for CAS is strategyproof for the colleges [8]; the proof makes use of the fact that the colleges do not (in general) have unit demand. It follows that no Pareto-stable mechanism for CAW is strategyproof for the colleges. Throughout the remainder of the paper, when we say that a mechanism for a college admissions model is strategyproof, we mean that it is strategyproof for the students.

Gale and Shapley's DA algorithm generalizes easily to the CAS model. Roth [8] has shown that the student-proposing DA algorithm provides a strategyproof stable mechanism for CAS when the preferences of the colleges are *responsive*. When the colleges have responsive preferences, the student-proposing DA mechanism is also known to be student-optimal for CAS [8].

Erdil and Ergin [2] consider the special case of the CAW model where the following restrictions hold for all students x and colleges y: x is not indifferent between being assigned to y and being left unassigned; y is not indifferent between having one of its slots assigned to x and having that slot left unfilled. We remark that this special case of CAW corresponds to the HRT problem discussed in Manlove [6, Chapter 3].[1] For this special case, Erdil and Ergin present a polynomial-time algorithm for computing a Pareto-stable matching when the preferences of the colleges satisfy a technical restriction related to responsiveness. We consider the same class of preferences, which we refer to as *minimally responsive*; see Sect. 5 for a formal definition. The algorithm of Erdil and Ergin does not provide a strategyproof mechanism. Chen and Ghosh [1] build on the results of Erdil and Ergin by considering the many-to-many generalization of HRT in which the agents on both sides of the market have capacities (and the preferences of any agent are minimally responsive). For this generalization, Chen and Ghosh provide a *strongly* polynomial-time algorithm. No strategyproof mechanism (even for the agents on one side of the market) is possible in the many-to-many setting. We provide the first Pareto-stable mechanism for CAW that is shown to be strategyproof. As in the work of Erdil-Ergin and Chen-Ghosh, we assume that the restriction preferences of the colleges

[1] In the model of Erdil and Ergin, which is stated using worker-firm terminology rather than student-college terminology, a "no indifference to unemployment/vacancy" assumption makes the aforementioned restrictions explicit. In the HRT model of Manlove, which is stated using resident-hospital terminology rather than student-college terminology, it is assumed that a set of acceptable resident-hospital pairs is given, and that each agent specifies weak preferences over the set of agents with whom they form an acceptable pair. We consider the approach of Erdil-Ergin — where the starting point is the preferences of the individual agents, and the "acceptability" of a given pair of agents may be deduced from those preferences — to be more natural, but the resulting models are equivalent.

are minimally responsive. We can also handle the class of college preferences "induced by additive utility" that is defined in Sect. 5.3.

The assignment game of Shapley and Shubik [10] can be viewed as an auction with multiple distinct items where each bidder is seeking to acquire at most one item. This class of *unit-demand auctions* has been heavily studied in the literature (see, e.g., Roth and Sotomayor [9, Chapter 8]). In Sect. 2, we define the notion of a "unit-demand auction with priorities" (UAP) and establish a number of useful properties of UAPs; these are straightforward generalizations of corresponding properties of unit-demand auctions. Section 3 builds on the UAP notion to define the notion of an "iterated UAP" (IUAP), and establishes a number of important properties of IUAPs; these results are nontrivial to prove and provide the technical foundation for our main results. Section 4 presents our first main result, a polynomial-time algorithm for SMIW that provides a strategyproof Pareto-stable mechanism. Section 5 presents our second main result, a polynomial-time algorithm for CAW that provides a strategyproof Pareto-stable mechanism assuming that the preferences of the colleges are minimally responsive.

2 Unit-Demand Auctions with Priorities

In this section, we formally define the notion of a unit-demand auction with priorities (UAP) and we establish Lemma 4, which is useful for analyzing the matching produced by the MatchSmiw algorithm to be presented in Sect. 4. Lemma 4 is used to establish Pareto-optimality (Lemma 28). In Sect. 2.1, we briefly discuss how to efficiently compute a desired matching in a UAP. In Sect. 2.2, we introduce a key definition that is helpful in establishing our strategyproofness results. We start with some useful definitions.

Each item v has an associated real reserve price, denoted $\mathit{reserve}(v)$. (We allow the reserve price to be negative to support procurement-style auction items.)

A *(unit-demand) bid β for a set of items V* is a subset of $V \times \mathbb{R}$ such that no two pairs in β share the same first component. (So β may be viewed as a partial function from V to $\mathbb{R}$.)

A *bidder u for a set of items V* is a triple (α, β, z) where α is an integer ID, β is a bid for V, and z is a real priority. If α is negative, then β is required to be of the form $\{(v, \mathit{reserve}(v))\}$ for some item v in V, and u is said to be a *reserve bidder for v*. For any bidder $u = (\alpha, \beta, z)$, we define $\mathit{id}(u)$ as α, $\mathit{bid}(u)$ as β, $\mathit{priority}(u)$ as z, and $\mathit{items}(u)$ as the union, over all (v, x) in β, of $\{v\}$.

A *unit-demand auction with priorities (UAP)* is a pair $A = (U, V)$ satisfying the following conditions: V is a set of items; U is a set of bidders for V; each bidder in U has a distinct ID; U contains exactly $|V|$ reserve bidders, one for each item v in V.

At times we view a UAP $A = (U, V)$ as an edge-weighted bipartite graph, where the set of edges incident on bidder u correspond to $\mathit{bid}(u)$: for each pair (v, x) in $\mathit{bid}(u)$, there is an edge (u, v) of weight x. We refer to a matching (resp., MCM, MWMCM) in the associated edge-weighted bipartite graph as a matching (resp., MCM, MWMCM) of A. For any edge $e = (u, v)$ in a given UAP, the associated weight is denoted $w(e)$ or $w(u, v)$. For any set of edges E, we define $w(E)$ as $\sum_{e \in E} w(e)$. For any UAP A, we let $w(A)$ denote the weight of an MWMCM of A.

Lemma 1. Let $A = (U, V)$ be a UAP, and let $\mathcal{I}$ denote the set of all subsets U' of U such that there exists an MWMCM of A that matches every bidder in U'. Then $(U, \mathcal{I})$ is a matroid.

Proof. The only nontrivial property to show is the exchange property. Assume that U_1 and U_2 belong to $\mathcal{I}$, $|U_1| > |U_2|$, and U_2 is not contained in U_1 . Let M_1 be an MWMCM of A that matches every bidder in U_1, and let M_2 be an MWMCM of A that matches every bidder in U_2. If M_2 matches some bidder u in $U_1 \setminus U_2$, then $U_2 + u$ belongs to $\mathcal{I}$, as required. Thus, in what follows, we assume that M_2 does not match any bidder in $U_1 \setminus U_2$. By considering the symmetric difference of M_1 and M_2, and making use of the fact that every MCM (and hence also every MWMCM) of A matches every item in V (due to the presence of the reserve bidders), it is straightforward to argue that there is a path P in (the edge-weighted bipartite graph associated with) A such that the following conditions hold: P has an even number of edges; the edges of P alternate between edges that are matched in M_1 and edges that are matched in M_2; one endpoint of P is a bidder u that belongs to U_1 (and hence is matched in M_1) and does not belong to U_2; the other endpoint of P is a bidder u' that does not belong to either U_1 or U_2. Let X_1 (resp., X_2) denote the set of all edges in P that belong to M_1 (resp., M_2). Let M denote $(M_2 \cup X_1) \setminus X_2$, which is easily seen to be an MCM of A. Since M_1 and M_2 are MWMCMs, we deduce that $w(X_1) = w(X_2)$, and hence that M is an MWMCM of A. Moreover, M matches $U_2 + u$, where u belongs to $U_1 \setminus U_2$, thereby establishing the exchange property. □

For any UAP A, we define $\mathit{matroid}(A)$ as the matroid of Lemma 1.

For any UAP $A = (U, V)$ and any independent set U' of $\mathit{matroid}(A)$, we define the *priority of* U' as the sum, over all bidders u in U', of $\mathit{priority}(u)$. For any UAP A, the matroid greedy algorithm can be used to compute a maximum-priority maximal independent set of $\mathit{matroid}(A)$.

For any matching M of a UAP $A = (U, V)$, we define $\mathit{matched}(M)$ as the set of all bidders in U that are matched in M. We say that an MWMCM M of a UAP A is *greedy* if $\mathit{matched}(M)$ is a maximum-priority maximal independent set of $\mathit{matroid}(A)$. For any UAP A, we define the predicate $\mathit{unique}(A)$ to hold if $\mathit{matched}(M) = \mathit{matched}(M')$ for all greedy MWMCMs M and M' of A.

For any matching M of a UAP, we define the *priority of* M, denoted $\mathit{priority}(M)$, as the sum, over all bidders u in $\mathit{matched}(M)$, of $\mathit{priority}(u)$. Thus an MWMCM is greedy if and only if it is a maximum-priority MWMCM.

Lemma 2. All greedy MWMCMs of a given UAP have the same distribution of priorities.

Proof. This is a standard matroid result that follows easily from the exchange property and the correctness of the matroid greedy algorithm. □

For any UAP A and any real priority z, we define $\mathit{greedy}(A, z)$ as the (uniquely defined, by Lemma 2) number of matched bidders with priority z in any greedy MWMCM of A.

Lemma 3. Let $A = (U, V)$ be a UAP. Let u be a bidder in U such that (v, x) belongs to $\mathit{bid}(u)$, $\mathit{priority}(u) = z$, and u is not matched in any greedy MWMCM of A. Let u' be a bidder in U such that (v, x') belongs to $\mathit{bid}(u')$, $\mathit{priority}(u') = z'$, and u' is matched to v in some greedy MWMCM of A. Then $(x, z) < (x', z')$.

Proof. Let M be a greedy MWMCM in which u' is matched to v. Thus u is not matched in M. Let M' denote $M - (u', v) + (u, v)$, which is an MCM of A. Since M is an MWMCM of A and $w(M') = w(M) - x' + x$, we conclude that $x \leq x'$. If $x < x'$, the claim of the lemma follows. Assume that $x = x'$. In this case, M' is an MWMCM of A since $w(M') = w(M)$. Since M is a greedy MWMCM of A and $\mathit{priority}(M') = \mathit{priority}(M) - z' + z$, we conclude that $z \leq z'$. If $z = z'$ then M' is a greedy MWMCM of A that matches u, a contradiction. Hence $z < z'$, as required. $\square$

Lemma 4. Let $A = (U, V)$ be a UAP, let M be a greedy MWMCM of A, let M' be an MWMCM of A, and let u' be a bidder that is matched in M' and unmatched in any greedy MWMCM of A. Then there is a bidder u such that u is matched in M, u is unmatched in M', and $\mathit{priority}(u) > \mathit{priority}(u')$.

Proof. It is straightforward to argue that the symmetric difference of M and M' includes a nonempty collection of even-length paths, each of which begins and ends at a bidder. It follows that u' is an endpoint of one such path, call it P. We claim that the other endpoint of P is a valid choice for the desired bidder u. Since P is of even length, we deduce that u is matched in M and unmatched in M'. It remains to argue that $\mathit{priority}(u) > \mathit{priority}(u')$. Let X denote the set of edges of P that belong to M (and not M'), and let X' denote the remaining edges of P, which belong to M' (and not M). Let M_0 denote $(M \cup X') \setminus X$. Since M_0 is an MCM of A and M is an MWMCM of A, we deduce that $w(X') \leq w(X)$. Let M_0' denote $(M' \cup X) \setminus X'$. Since M_0' is an MCM of A and M' is an MWMCM of A, we deduce that $w(X) \leq w(X')$. Thus $w(X) = w(X')$, and we conclude that M_0 and M_0' are MWMCMs of A. Since M is a greedy MWMCM of A and M_0 is an MWMCM of A, we deduce that $\mathit{priority}(u) \geq \mathit{priority}(u')$. If $\mathit{priority}(u) = \mathit{priority}(u')$, then we find that M_0 is a greedy MWMCM of A (since M is a greedy MWMCM of A), a contradiction since u' is matched in M_0. Thus $\mathit{priority}(u) > \mathit{priority}(u')$, as required. $\square$

2.1 Finding a Greedy MWMCM

In this section, we briefly discuss how to efficiently compute a greedy MWMCM of a UAP via a slight modification of the classic Hungarian method for the assignment problem [5]. In the (maximization version of the) assignment problem, we are given a set of n agents, a set of n tasks, and a weight for each agent-task pair, and our objective is to find a perfect matching (i.e., every agent and task is required to be matched) of maximum total weight. The Hungarian method for the assignment problem proceeds as follows: a set of dual variables, namely a "price" for each task, and a possibly incomplete matching are maintained; an arbitrary unmatched agent u is chosen and a shortest augmenting path from u to an unmatched task is computed using "residual costs" as the edge weights; an augmentation is performed along the path to update the matching, and the dual variables are adjusted accordingly to maintain complementary slackness; the process repeats until a perfect matching is found.

Within our UAP setting, the set of bidders can be larger than the set of items, and some bidder-item pairs may not be matchable, i.e., the associated bipartite graph is not necessarily complete. In this setting, we can use an "incremental" version of the Hungarian method to find an (not necessarily greedy) MWMCM of a given UAP $A = (U, V)$ as follows. We start with the MCM M that matches each item v in V to the reserve bidder in U for v. Then, for each non-reserve bidder u in U (in arbitrary order), we process u via an "incremental Hungarian step" as follows: find the shortest paths from u to each bidder in $\mathit{matched}(M) + u$ (including the path of length zero from u to u) in the residual graph; let W denote the minimum path weight among these shortest paths; identify the nonempty set U' of bidders in $\mathit{matched}(M) + u$ that can be reached from u via a shortest path of weight W; choose an arbitrary bidder u' from U'; augment M along the shortest path from u to u', i.e., if $u \neq u'$ then u' becomes unmatched and u becomes matched; adjust the prices accordingly to maintain complementary slackness; update the residual graph. The algorithm terminates when every non-reserve bidder has been processed. The algorithm performs $|U|$ incremental Hungarian steps and each incremental Hungarian step can be implemented in $O(|V| \log |V| + m)$ time by utilizing Fibonacci heaps [3], where m denotes the number of edges in the residual graph, which is $O(|V|^2)$.

In order to find a greedy MWMCM, we slightly modify the implementation described in the previous paragraph. Lemma 36 established in App. A implies that if we choose a bidder u' having minimum priority among the ones in U' (instead of choosing an arbitrary one) at each incremental Hungarian step, then the algorithm described above outputs a greedy MWMCM. It is easy to see that the added cost of selecting a minimum priority bidder at each iteration does not increase the asymptotic time complexity of the algorithm.

2.2 Threshold of an Item

In this section, we define the notion of a "threshold" of an item in a UAP. This lays the groundwork for a corresponding IUAP definition in Sect. 3.2. Item thresholds play an important role in our strategyproofness results. We start with some useful definitions.

Let $A = (U, V)$ be a UAP and let u be a bidder such that $id(u)$ is nonnegative and is not equal to the ID of any bidder in U. Then we define $A + u$ as the UAP $(U + u, V)$.

For any UAPs $A = (U, V)$ and $A' = (U', V')$, we say that A' extends A if $U \subseteq U'$ and $V = V'$.

Lemma 5. Let $A = (U, V)$ be a UAP, let u be a bidder in U that is not matched in any greedy MWMCM of A, and let $A' = (U', V)$ be a UAP that extends A. Then u is not matched in any greedy MWMCM of A'.

Proof. Assume for the sake of contradiction that there is a greedy MWMCM M' of A' such that u is matched in M'. Let M be a greedy MWMCM of A minimizing $|M \oplus M'|$. The symmetric difference of M and M' contains an even-length path P from u to some bidder u' that is matched in M. Since all of the vertices on this path belong to U, we can get a matching M'' of A by starting with M and exchanging along this path (take the M' edges instead of the M edges). It is easy to argue that the M-edges on P have the same total weight as the M'-edges on P, and u and u' have the same priority. It follows that M'' is a greedy MWMCM of A that matches u, a contradiction. □

Lemma 6. Let $A = (U, V)$ be a UAP and let v be an item in V. Let U' be the set of bidders u such that $A + u$ is a UAP and $bid(u)$ is of the form $\{(v, x)\}$. Then there is a unique pair of reals (x^*, z^*) such that for any bidder u in U', the following conditions hold, where A' denotes $A + u$, x denotes $w(u, v)$, and z denotes $priority(u)$: (1) if $(x, z) > (x^*, z^*)$ then u is matched in every greedy MWMCM of A'; (2) if $(x, z) < (x^*, z^*)$ then u is not matched in any greedy MWMCM of A'; (3) if $(x, z) = (x^*, z^*)$ then u is matched in some but not all greedy MWMCMs of A'.

Proof. Let M be a greedy MWMCM of A, let W denote $w(M)$, and let Z denote $priority(M)$. Let $\mathcal{M}$ denote the set of matchings of A' with cardinality $|V| - 1$ that do not match v, let $\mathcal{M}'$ denote the maximum-weight elements of $\mathcal{M}$, let $\mathcal{M}''$ denote the maximum-priority elements of $\mathcal{M}'$, and observe that there is a unique pair of reals (W', Z') such that any matching M' in $\mathcal{M}''$ has weight

W' and priority Z'. It is straightforward to verify that the unique choice of (x^*, z^*) satisfying the conditions stated in the lemma is $(W - W', Z - Z')$. □

For any UAP $A = (U, V)$ and any item v in V, we define the unique pair (x^*, z^*) of Lemma 6 as $\mathit{threshold}(A, v)$.

3 Iterated Unit-Demand Auctions with Priorities

In this section, we formally define the notion of an iterated unit-demand auction with priorities (IUAP). An IUAP allows the bidders, called "multibidders" in this context, to have a sequence of unit-demand bids instead of a single unit-demand bid. In Sect. 3.1, we define a mapping from an IUAP to a UAP by describing an algorithm that generalizes the DA algorithm, and we establish two lemmas that are useful for analyzing the matching produced by the MATCHSMIW algorithm to be presented in Sect. 4. Lemmas 11 and 12 are used to establish weak stability (Lemmas 25, 26, and 27), and Lemma 12 is used to establish Pareto optimality (Lemma 28). In Sect. 3.2, we define the threshold of an item in an IUAP and we establish Lemma 15, which plays a key role in establishing our strategyproofness results. We start with some useful definitions.

A *multibidder t for a set of items V* is a pair (σ, z) where z is a real priority and σ is a sequence of bidders for V such that the following conditions hold: all of the bidders in the sequence have distinct IDs and a common priority z; if a bidder u in the sequence is a reserve bidder for some item v, then σ is equal to $\langle u \rangle$, and we say that t is a reserve multibidder for item v. We define $\mathit{priority}(t)$ as z. For any integer i such that $1 \leq i \leq |\sigma|$, we define $\mathit{bidder}(t, i)$ as the bidder $\sigma(i)$. For any integer i such that $0 \leq i \leq |\sigma|$, we define $\mathit{bidders}(t, i)$ as $\{\mathit{bidder}(t, j) \mid 1 \leq j \leq i\}$. We define $\mathit{bidders}(t)$ as $\mathit{bidders}(t, |\sigma|)$.

An *iterated UAP (IUAP)* is a pair $B = (T, V)$ where V is a set of items and T is a set of multibidders for V that contains exactly $|V|$ reserve multibidders, one for each item v in V. In addition, for any distinct multibidders t and t' in T, the following conditions hold: $\mathit{priority}(t) \neq \mathit{priority}(t')$; if u belongs to $\mathit{bidders}(t)$ and u' belongs to $\mathit{bidders}(t')$, then $\mathit{id}(u) \neq \mathit{id}(u')$.

For any IUAP $B = (T, V)$ and any item v in V, we define $\mathit{dummy}(B, v)$ as $\mathit{bidder}(t, 1)$ where t is the unique reserve multibidder for v in T, and we define $\mathit{dummies}(B)$ as $\{\mathit{dummy}(B, v) \mid v \in V\}$.

For any IUAP $B = (T, V)$, we define $\mathit{bidders}(B)$ as the union, over all t in T, of $\mathit{bidders}(t)$.

3.1 Mapping an IUAP to a UAP

Having defined the notion of an IUAP, we now describe an algorithm TOUAP that maps a given IUAP to a UAP. Algorithm TOUAP generalizes the DA algorithm. In each iteration of the DA algorithm, a single man is nondeterministically chosen, and this man reveals his next choice. In each iteration of TOUAP, a single multibidder is nondeterministically chosen, and this multibidder reveals its next bid. We prove in Lemma 10 that, like the DA algorithm, algorithm TOUAP is confluent: the output does not depend on the nondeterministic choices during an execution. We conclude this section by establishing Lemmas 11 and 12, which are useful for analyzing the matching produced by algorithm MATCHSMIW in Sect. 4. Lemmas 11 and 12 are used to establish weak stability (Lemmas 25, 26, and 27), and Lemma 12 is used to establish Pareto optimality (Lemma 28). We start with some useful definitions.

Let A be a UAP (U, V) and let B be an IUAP (T, V). The predicate $\mathit{prefix}(A, B)$ is said to hold if $U \subseteq \mathit{bidders}(B)$ and for any multibidder t in T, $U \cap \mathit{bidders}(t) = \mathit{bidders}(t, i)$ for some i.

A *configuration* C is a pair (A, B) where A is a UAP, B is an IUAP, and $\mathit{prefix}(A, B)$ holds.

Let $C = (A, B)$ be a configuration, where $A = (U, V)$ and $B = (T, V)$, and let u be a bidder in U. Then we define $\mathit{multibidder}(C, u)$ as the unique multibidder t in T such that u belongs to $\mathit{bidders}(t)$.

Let $C = (A, B)$ be a configuration where $A = (U, V)$ and $B = (T, V)$. For any t in T, we define $\mathit{bidders}(C, t)$ as $\{u \in U \mid \mathit{multibidder}(C, u) = t\}$.

Let $C = (A, B)$ be a configuration where $B = (T, V)$. We define $\mathit{ready}(C)$ as the set of all bidders u in $\mathit{bidders}(B)$ such that $\mathit{greedy}(A, \mathit{priority}(u)) = 0$ and $u = \mathit{bidder}(t, |\mathit{bidders}(C, t)| + 1)$ where $t = \mathit{multibidder}(C, u)$.

Algorithm 1 ToUap(B)

Input: An IUAP $B = (T, V)$

$A \leftarrow (\mathit{dummies}(B), V)$
$C \leftarrow (A, B)$
while $\mathit{ready}(C)$ is nonempty **do**
  $A \leftarrow A +$ an arbitrary bidder in $\mathit{ready}(C)$
  $C \leftarrow (A, B)$
end while
return A

Our algorithm that maps an IUAP to a UAP is given in Alg. 1. The input is an IUAP B and the output is a UAP A such that $\mathit{prefix}(A, B)$ holds. The algorithm starts with the UAP consisting of the $|V|$ reserve bidders in B and all the items in V. At this point, none of the bidders associated with the non-reserve bidders have been "revealed", i.e., no non-reserve bidder is present in the initial UAP. Then, the algorithm iteratively and nondeterministically chooses a "ready" bidder and "reveals" it by adding it to the UAP that is maintained in the program variable A. A bidder u associated with some multibidder $t = (\sigma, z)$ is ready if u is not revealed and for each bidder u' that precedes u in σ, u' is revealed and it is not matched in any greedy MWMCM of A. It is easy to verify that the predicate $\mathit{prefix}(A, B)$ is an invariant of the algorithm loop: If a bidder u belonging to a multibidder t is to be revealed at an iteration and $U \cap \mathit{bidders}(t) = \mathit{bidders}(t, i)$ for some integer i at the beginning of this iteration, then $U \cap \mathit{bidders}(t) = \mathit{bidders}(t, i + 1)$ after revealing u, where (U, V) is the UAP that is maintained by the program variable A at the beginning of the iteration; it is trivial to verify the conditions regarding the multibidders other than t. It is also easy to see that the algorithm terminates, because no bidder can be revealed more than once since a bidder cannot be ready after it has been revealed. We now argue that the output of the algorithm is uniquely determined (Lemma 10), even though the bidder that is revealed at each iteration is chosen nondeterministically.

For any configuration $C = (A, B)$, we define the predicate $\mathit{tail}(C)$ to hold if for any bidder u that is matched in some greedy MWMCM of A, we have $u = \mathit{bidder}(t, |\mathit{bidders}(C, t)|)$ where t denotes $\mathit{multibidder}(C, u)$.

Lemma 7. Let $C = (A, B)$ be a configuration where $B = (T, V)$ and assume that $\mathit{tail}(C)$ holds. Then $\mathit{greedy}(A, \mathit{priority}(t)) \leq 1$ for each t in T.

Proof. The claim of the lemma easily follows from the definition of $tail(C)$. □

Lemma 8. The predicate $tail(C)$ is an invariant of the Alg. 1 loop.

Proof. It is easy to see that $tail(C)$ holds when the loop is first encountered. Now consider an iteration of the loop that takes us from configuration $C = (A, B)$ where $A = (U, V)$ to configuration $C' = (A', B)$ where $A' = (U', V)$. We need to show that $tail(C')$ holds. Let u be a bidder that is matched in some greedy MWMCM M' of A'. Let t denote $multibidder(C', u)$. If u belongs to $dummies(B)$, then $|bidders(t)| = |bidders(C', t)| = 1$ and $u = bidder(t, 1)$, as required. For the remainder of the proof, assume that u does not belong to $dummies(B)$. Let u^* denote the bidder that is added to A in line 4, and consider the following three cases.

Case 1: $u = u^*$. Let t denote $multibidder(C', u^*)$. In this case, $|bidders(C, t)| + 1 = |bidders(C', t)|$, so $u^* = bidder(t, |bidders(C', t)|)$, as required.

Case 2: $u \neq u^*$ and $priority(u) \neq priority(u^*)$. Since U' contains U, Lemma 5 implies that u is matched in some greedy MWMCM of A. Since C is a configuration and $tail(C)$ holds, we deduce that $u = bidder(t, |bidders(C, t)|)$ where t denotes $multibidder(C, u)$. Since $multibidder(C', u) = multibidder(C, u)$ and $bidders(C', t) = bidders(C, t)$, we conclude that $u = bidder(t, |bidders(C', t)|)$ where t denotes $multibidder(C', u)$, as required.

Case 3: $u \neq u^*$ and $priority(u) = priority(u^*)$. Since u^* belongs to $ready(C)$, we know that $greedy(A, priority(u)) = 0$. Also, since u is not u^*, u belongs to U and we conclude that u is not matched in any greedy MWMCM of A. Since U' contains U, Lemma 5 implies that u is not matched in any greedy MWMCM of A', a contradiction. □

Lemma 9. Let $C = (A, B)$ be a configuration such that $tail(C)$ holds. Then $unique(A)$ holds.

Proof. Let M and M' be greedy MWMCMs of A, and let u be a bidder in $matched(M)$. To establish the lemma, it is sufficient to prove that u belongs to $matched(M')$. Let t denote $multibidder(C, u)$ and let z denote $priority(t)$. Since $tail(C)$ holds, we know that $u = bidder(t, |bidders(C, t)|)$. Since u is matched by M and since $tail(C)$ holds, Lemma 7 implies that $greedy(A, z) = 1$. Thus Lemma 2 implies that M' matches one priority-z bidder, which is equal to u since $tail(C)$ holds. □

Lemma 10. Let $B = (T, V)$ be an IUAP. Then all executions of Alg. 1 on input B produce the same output.

Proof. Suppose not, and let X_1 and X_2 denote two executions of Alg. 1 on input B that produce distinct output UAPs $A_1 = (U_1, V)$ and $A_2 = (U_2, V)$. Without loss of generality, assume that $|U_1| \geq |U_2|$. Then there is a first iteration of execution X_1 in which the bidder added to A in line 4 belongs to $U_1 \setminus U_2$; let u' denote this bidder. Let $C' = (A', B)$ where $A' = (U', V)$ denote the configuration in program variable C at the start of this iteration, and let t' denote $multibidder(C', u')$. Let i be the integer such that $u' = bidder(t', i)$. We know that $i > 1$ because it is easy to see that U_2 contains $bidder(t', 1)$. Let u'' denote $bidder(t', i - 1)$. Since u' belongs to $ready(C')$, Lemmas 8 and 9 imply that u'' is not matched in any greedy MWMCM of A'. Since U' is contained in U_2, Lemma 5 implies that u'' is not matched in any greedy MWMCM of A_2. Let $C_2 = (A_2, B)$ denote the final configuration of execution X_2; thus $ready(C_2)$ is empty and $|bidders(C_2, t')| = i - 1$. By Lemma 8, we conclude that $greedy(A_2, priority(t')) = 0$, and hence that u'' is contained in $ready(C_2)$, a contradiction. □

For any IUAP B, we define $uap(B)$ as the unique (by Lemma 10) UAP returned by any execution of Alg. 1 on input B.

We can incorporate the modified incremental Hungarian step of Sect. 2.1 into each iteration of the loop of Alg. 1 in order to both obtain an efficient implementation and to extend the algorithm so that it also maintains and returns a greedy MWMCM of the UAP A as follows: we maintain dual variables (a price for each item) and a residual graph; the initial greedy MWMCM is the matching that matches the reserve bidders to the items; when a bidder u is added to A at line 4, we perform an incremental Hungarian step to process u to update the greedy MWMCM, the prices, and the residual graph. Since we maintain a greedy MWMCM of A at each iteration of the loop, it is easy to see that identifying a bidder in $ready(C)$ (or determining that this set is empty) takes $O(|V|)$ time. Thus the whole algorithm can be implemented in $O(|bidders(B)| \cdot |V|^2)$ time.

We now present two lemmas that are used in Sect. 4: Lemmas 11 and 12 are used to establish weak stability (Lemmas 25, 26, and 27); Lemma 12 is used to establish Pareto optimality (Lemma 28).

Lemma 11. Let B be an IUAP and let $uap(B)$ be $A = (U, V)$. Let u be a bidder in U and let M be a greedy MWMCM of A such that (v, x) belongs to $bid(u)$ and u is not matched in M. Let u' be a bidder in U such that (v, x') belongs to $bid(u')$ and u' is matched to v in some greedy MWMCM of A. Then $(x, z) < (x', z')$, where z denotes $priority(u)$ and z' denotes $priority(u')$.

Proof. Since u is not matched in M and since Lemmas 8 and 9 imply that $unique(A)$ holds, we deduce that u is not matched in any greedy MWMCM of A. The claim follows from Lemma 3. □

Lemma 12. Let $B = (T, V)$ be an IUAP, let (σ, z) be a multibidder that belongs to T, let $uap(B)$ be (U, V), and let M be a greedy MWMCM of the UAP (U, V). Then the following claims hold: (1) if $\sigma(k)$ is matched in M for some k, then $\sigma(k') \in U$ if and only if $1 \le k' \le k$; (2) if $\sigma(k)$ is not matched in M for any k, then $\sigma(k) \in U$ for $1 \le k \le |\sigma|$.

Proof. Let X denote an execution of Alg. 1 on input B. For the first claim, assume that $\sigma(k)$ is matched in M for some k. It is easy to see that Alg. 1 adds $\sigma(k)$ to $uap(B)$ after adding $\sigma(1), \ldots, \sigma(k-1)$. We now claim that no bidder $\sigma(k')$ for $k < k' \le |\sigma|$ belongs to U. Suppose the claim does not hold, and let k' be an integer such that $k < k'$ and $\sigma(k')$ belongs to U. Consider the iteration in X in which bidder $\sigma(k')$ is added to A in line 4, and let $C' = (A', B)$ denote the state of the program variable C at the beginning of this iteration. Since $\sigma(k')$ belongs to $ready(C')$, we deduce that $\sigma(k)$ belongs to A', $greedy(A', z) = 0$, and hence $\sigma(k)$ is not matched in any greedy MWMCM of A'. Then, since $uap(B)$ extends A', Lemma 5 implies that $\sigma(k)$ is not matched in any greedy MWMCM of $uap(B)$, a contradiction.

For the second claim, assume that $\sigma(k)$ is not matched in M for any k. For the sake of contradiction, assume that $\sigma(k)$ does not belong to U for some k, and let k' denote smallest such k. Consider the time in X when the **while** loop terminates, i.e., when the condition of the loop fails because $ready(C)$ is empty. Since $\sigma(k)$ is not matched in M for any k, we deduce that $greedy(A, z) = 0$. Then, since $\sigma(1), \ldots, \sigma(k'-1)$ belongs to U, we deduce that $\sigma(k')$ belongs to $ready(C)$, a contradiction. □

3.2 Threshold of an Item

In this section, we define the threshold of an item in an IUAP and we establish Lemma 15, which plays a key role in establishing our strategyproofness results. We start with some useful definitions.

For any IUAP B, Lemmas 8 and 9 imply that $unique(uap(B))$ holds, and thus that every greedy MWMCM of $uap(B)$ matches the same set of bidders. We define this set of matched bidders as $winners(B)$. For any IUAP B, we define $losers(B)$ as $U \setminus winners(B)$ where (U, V) is $uap(B)$.

Let $B = (T, V)$ be an IUAP and let $u = (\alpha, \beta, z)$ be a bidder for V such that the following conditions hold: α is nonnegative and is not equal to the ID of any bidder in $bidders(B)$; z is not equal to the priority of any bidder in $dummies(B)$. Then we define the IUAP $B + u$ as follows: if T contains a multibidder t of the form (σ, z) for some sequence of bidders σ, then we define $B + u$ as $(T - t + t', V)$ where $t' = (\sigma', z)$ and σ' is the sequence of bidders obtained by appending u to σ; otherwise, we define $B + u$ as $(T + t, V)$ where $t = (\langle u \rangle, z)$.

Lemma 13. Let $B = (T, V)$ and $B' = B + u$ be IUAPs. Then $losers(B) \subseteq losers(B')$.

Proof. Let u' be a bidder in $losers(B)$. Thus u' is not matched in any greedy MWMCM of $uap(B)$. Using Lemma 10, it is easy to see that $uap(B')$ extends $uap(B)$. Thus Lemma 5 implies that u' is not matched in any greedy MWMCM of $uap(B')$, and hence that u' belongs to $losers(B')$. □

Lemma 14. Let $B = (T, V)$ be an IUAP and let v be an item in V. For $i \in \{1, 2\}$, let $B_i = B + u_i$ be an IUAP where $u_i = (\alpha_i, \{(v, x_i)\}, z_i)$. Let $A_1 = (U_1, V)$ denote $uap(B_1)$ and let $A_2 = (U_2, V)$ denote $uap(B_2)$. Assume that $\alpha_1 \neq \alpha_2$, $z_1 \neq z_2$, and u_1 belongs to $winners(B_1)$. Then the following claims hold: if u_2 belongs to $winners(B_2)$ then $U_1 - u_1 = U_2 - u_2$; if u_2 belongs to $losers(B_2)$ then $U_1 - u_1$ contains $U_2 - u_2$.

Proof. Let B_3 denote the IUAP $B_1 + u_2$ which is equal to the IUAP $B_2 + u_1$. For the first claim, assume that u_2 belongs to $winners(B_2)$. Using Lemma 10, it is straightforward to argue that $uap(B_3)$ is equal to $A_1 + u_2 = (U_1 + u_2, V)$ and is also equal to $A_2 + u_1 = (U_2 + u_1, V)$. Since u_1 belongs to U_1 and u_2 belongs to U_2, we conclude that $U_1 - u_1 = U_2 - u_2$, as required.

For the second claim, assume that u_2 belongs to $losers(B_2)$. Suppose $(x_1, z_1) < (x_2, z_2)$. Then Lemmas 10 and 3 imply that u_2 belongs to $winners(B_3)$. But since u_2 belongs to $losers(B_2)$, Lemma 13 implies that u_2 belongs to $losers(B_2{+}u_1) = losers(B_3)$, a contradiction. Since $z_1 \neq z_2$, we conclude that $(x_1, z_1) > (x_2, z_2)$. Then, Lemma 10 implies that $uap(B_3) = uap(B_1) + u_2 = (U_1 + u_2, V)$. Since Lemma 10 also implies that $uap(B_3)$ extends $uap(B_2)$, it follows that $U_1 + u_2$ contains U_2, and hence that U_1 contains $U_2 - u_2$. Since u_1 does not belong to U_2, we conclude that $U_1 - u_1$ contains $U_2 - u_2$, as required. □

We are now ready to define the threshold of an item in an IUAP, and to state Lemma 15. In Sect. 4, Lemma 15 plays an important role in establishing that our SMIW mechanism is strategyproof (Lemma 30). The proof of Lemma 15 is provided in Sect. 3.2.1.

Let $B = (T, V)$ be an IUAP and let v be an item in V. By Lemma 14, there is a unique subset U of $bidders(B)$ such that the following condition holds: for any IUAP $B' = B + u$ where u is of the form $(\alpha, \{(v, x)\}, z)$ and u belongs to $winners(B')$, $uap(B')$ is equal to $(U + u, V)$. We define $uap(B, v)$ as the UAP (U, V) and we define $threshold(B, v)$ as $threshold(uap(B, v), v)$.

Lemma 15. Let $B = (T, V)$ be an IUAP, let $t = (\sigma, z)$ be a non-reserve multibidder that belongs to T, and let B' denote the IUAP $(T - t, V)$. Suppose that $(\sigma(k), v)$ is matched in some greedy MWMCM of $uap(B)$ for some k. Then, we have

$$(w(\sigma(k), v), z) \geq \mathit{threshold}(B', v). \tag{1}$$

Furthermore, for each k' and v' such that $1 \leq k' < k$ and $v' \in \mathit{items}(\sigma(k'))$, we have

$$(w(\sigma(k'), v'), z) < \mathit{threshold}(B', v'). \tag{2}$$

3.2.1 Proof of Lemma 15

The sole purpose of this section is to prove Lemma 15. We do so by establishing a stronger result, namely Lemma 24 below. We start with a useful definition.

For any IUAP B, we define $\mathit{priorities}(B)$ as $\{z \mid u \in \mathit{winners}(B) \text{ and } \mathit{priority}(u) = z\}$.

Lemma 16. Let $B = (T, V)$ be an IUAP. Then $|\mathit{priorities}(B)| = |V|$.

Proof. Lemma 7 implies that distinct bidders in $\mathit{winners}(B)$ have distinct priorities; the claim of the lemma follows since $|\mathit{winners}(B)| = |V|$. □

Lemma 17. Let $B = (T, V)$ and $B' = B + u = (T', V)$ be IUAPs, let Z denote $\mathit{priorities}(B)$, let Z' denote $\mathit{priorities}(B')$, and let z denote $\mathit{priority}(u)$. Then $Z' \subseteq Z + z$.

Proof. Consider running Alg. 1 on input B', where we avoid selecting bidder u from $\mathit{ready}(C)$ unless it is the only bidder in $\mathit{ready}(C)$. (By Lemma 10, the final output is the same regardless of which bidder we select from $\mathit{ready}(C)$ at each iteration.) If u never enters $\mathit{ready}(C)$, then $uap(B') = uap(B)$, and so $Z' = Z$, and the claim of the lemma holds.

Now suppose that u does enter $\mathit{ready}(C)$ at some point. At the start of the iteration in which u is selected from $\mathit{ready}(C)$, the UAP $A = (U, V)$ is equal to $uap(B)$, so by Lemma 7, every greedy MWMCM of A matches exactly one bidder of each priority in Z. Furthermore, letting U' denote the set of all bidders u' in $\mathit{bidders}(B)$ such that $\mathit{priority}(u')$ does not belong to $Z + z$, we deduce that U' is contained in $\mathit{losers}(B) = U \setminus \mathit{winners}(B)$. Then Lemma 13 implies that no bidder in U' is matched in any greedy MWMCM of $uap(B')$, and the claim of the lemma follows. □

Lemma 18. Let $A = (U, V)$ and $A' = A + u$ be UAPs, and let v be an item in V. Then $\mathit{threshold}(A, v) \leq \mathit{threshold}(A', v)$.

Proof. Assume for the sake of contradiction that $\mathit{threshold}(A, v) > \mathit{threshold}(A', v)$. Then there exists a bidder u' such that u' does not belong to $U + u$, $\mathit{bid}(u') = \{(v, x)\}$, $\mathit{priority}(u') = z$, and

$$\mathit{threshold}(A', v) < (x, z) < \mathit{threshold}(A, v).$$

Since $(x, z) < \mathit{threshold}(A, v)$, Lemma 6 implies that u' is not matched in any greedy MWMCM of $A + u'$. Thus Lemma 5 implies that u' is not matched in any greedy MWMCM of $A' + u'$. On the other hand, since $\mathit{threshold}(A', v) < (x, z)$, Lemma 6 implies that u' is matched in any greedy MWMCM of $A' + u'$, a contradiction. □

Lemma 19. Let $B = (T, V)$ and $B' = B + u$ be IUAPs where $u = (\alpha, \{(v, x)\}, z)$, v is an item in V, and z does not belong to $\mathit{priorities}(B)$. If u belongs to $\mathit{winners}(B')$, then $(x, z) > \mathit{threshold}(B, v)$. If u belongs to $\mathit{losers}(B')$, then $(x, z) < \mathit{threshold}(B, v)$.

Proof. First, assume that u belongs to $\mathit{winners}(B')$. Thus u is matched in every greedy MWMCM of $\mathit{uap}(B')$, which is equal to $\mathit{uap}(B, v) + u$ by definition. Lemma 6 implies that $(x, z) > \mathit{threshold}(\mathit{uap}(B, v), v) = \mathit{threshold}(B, v)$, as required.

Now assume that u belongs to $\mathit{losers}(B')$. Thus u is not matched in any greedy MWMCM of $\mathit{uap}(B')$. Define U so that $\mathit{uap}(B') = (U + u, V)$, and let A denote the UAP (U, V). Lemma 6 implies that $(x, z) < \mathit{threshold}(A, v)$. Lemma 14 implies that $\mathit{uap}(B, v) + u$ extends $\mathit{uap}(B')$, and hence that $\mathit{uap}(B, v)$ extends A. Lemma 18 therefore implies that

$$\mathit{threshold}(A, v) \leq \mathit{threshold}(\mathit{uap}(B, v), v) = \mathit{threshold}(B, v).$$

Thus $(x, z) < \mathit{threshold}(B, v)$, as required. □

Lemma 20. Let $B = (T, V)$ and $B' = B + u$ be IUAPs, and let v be an item in V. Then $\mathit{threshold}(B, v) \leq \mathit{threshold}(B', v)$.

Proof. Let (x, z) denote $\mathit{threshold}(B, v)$ and let (x', z') denote $\mathit{threshold}(B', v)$, and assume for the sake of contradiction that $(x, z) > (x', z')$.

Let u' be a bidder $(\alpha, \{(v, x)\}, z'')$ such that z'' does not belong to $\mathit{priorities}(B) + \mathit{priority}(u)$, $z > z''$, and $(x, z'') > (x', z')$. Let B'' denote $B + u'$ and let B''' denote $B' + u'$. Since z'' does not belong to $\mathit{priorities}(B)$, we deduce that u' belongs to either $\mathit{winners}(B'')$ or $\mathit{losers}(B'')$. Then, by Lemma 19, u' belongs to $\mathit{losers}(B'')$, and hence by Lemma 13, u' belongs to $\mathit{losers}(B''')$. On the other hand, since z'' does not belong to $\mathit{priorities}(B) + \mathit{priority}(u)$, Lemma 17 implies that z'' does not belong to $\mathit{priorities}(B')$, and we deduce that u' belongs to either $\mathit{winners}(B''')$ or $\mathit{losers}(B''')$. Then, Lemma 19 implies that u' belongs to $\mathit{winners}(B''')$, a contradiction. □

Lemma 21. Let $B = (T, V)$ and $B' = B + u$ be IUAPs where u belongs to $\mathit{losers}(B')$, and let v be an item in V. Then $\mathit{threshold}(B', v) = \mathit{threshold}(B, v)$.

Proof. Suppose not. Then by Lemma 20, we have $\mathit{threshold}(B, v) < \mathit{threshold}(B', v)$. Let z denote $\mathit{priority}(u)$. Since $B' = B + u$ and u belongs to $\mathit{losers}(B')$, we deduce that z does not belong to $\mathit{priorities}(B)$. Since u belongs to $\mathit{losers}(B')$, we deduce that z does not belong to $\mathit{priorities}(B')$. Hence Lemmas 16 and 17 imply that $\mathit{priorities}(B') = \mathit{priorities}(B)$.

Let B'' denote $B + u'$ where $u' = (\alpha, \{(v, x')\}, z')$ is a bidder such that z' does not belong to $\mathit{priorities}(B) + z$ and $\mathit{threshold}(B, v) < (x', z') < \mathit{threshold}(B', v)$.

Let B''' denote $B' + u'$. Since z' does not belong to $\mathit{priorities}(B) + z$, Lemma 17 implies that z' does not belong to $\mathit{priorities}(B')$, and we deduce that u' belongs to either $\mathit{winners}(B''')$ or $\mathit{losers}(B''')$. Since $(x', z') < \mathit{threshold}(B', v)$, Lemma 19 implies that u' belongs to $\mathit{losers}(B''')$. Hence Lemmas 16 and 17 imply that $\mathit{priorities}(B''') = \mathit{priorities}(B')$. Since we have established above that $\mathit{priorities}(B') = \mathit{priorities}(B)$, we deduce that $\mathit{priorities}(B''') = \mathit{priorities}(B)$.

Since z' does not belong to $\mathit{priorities}(B)$, we deduce that u' belongs to either $\mathit{winners}(B'')$ or $\mathit{losers}(B'')$. Since $(x', z') > \mathit{threshold}(B, v)$, Lemma 19 implies that u' belongs to $\mathit{winners}(B'')$ and hence z' belongs to $\mathit{priorities}(B'')$. Hence Lemmas 16 and 17 imply that there exists a real z'' in $\mathit{priorities}(B)$ that does not belong to $\mathit{priorities}(B'')$. Since z does not belong to $\mathit{priorities}(B)$, we have $z \neq z''$. Since $B''' = B'' + u$ and $z \neq z''$, Lemma 17 implies that z'' does not belong to $\mathit{priorities}(B''')$, a contradiction since $\mathit{priorities}(B''') = \mathit{priorities}(B)$. □

Lemma 22. Let $B = (T, V)$ and $B' = B + u$ be IUAPs where $u = (\alpha, \beta, z)$ and z does not belong to $\mathit{priorities}(B)$, and let v be an item in V. Assume that (v, x) belongs to β, and that $\mathit{threshold}(B, v) < (x, z)$. Then u belongs to $\mathit{winners}(B')$.

Proof. Suppose not. Let $A' = (U', V)$ denote $\mathit{uap}(B')$. Since z does not belong to $\mathit{priorities}(B)$, we deduce that u belongs to U'. Thus u belongs to $U' \setminus \mathit{winners}(B') = \mathit{losers}(B')$, and so $\mathit{threshold}(B', v) = \mathit{threshold}(B, v)$ by Lemma 21.

Let B'' denote $B' + u'$ where $u' = (\alpha, \{(v, x)\}, z')$ is a bidder such that z' does not belong to $\mathit{priorities}(B) + z$, $\mathit{threshold}(B, v) < (x, z')$, and $z' < z$. Since z' does not belong to $\mathit{priorities}(B) + z$, we deduce that u' belongs to either $\mathit{winners}(B'')$ or $\mathit{losers}(B'')$. Then, by Lemma 19, u' belongs to $\mathit{winners}(B'')$. Let $A'' = (U'', V)$ denote $\mathit{uap}(B'')$, and let M be a greedy MWMCM of A''. Since u' belongs to $\mathit{winners}(B'')$, the edge (u', v) belongs to M. Since u belongs to $\mathit{losers}(B')$, Lemma 13 implies that u belongs to $\mathit{losers}(B'')$, and hence that u is unmatched in M. By Lemma 3, we find that $(x, z) < (x, z')$ and hence $z < z'$, a contradiction. □

Lemma 23. Let $B = (T, V)$ and $B_0 = B + u$ be IUAPs where $u = (\alpha, \beta, z)$, z does not belong to $\mathit{priorities}(B)$, and $\beta = \{(v_1, x_1), \ldots, (v_k, x_k)\}$. Assume that $(x_i, z) < \mathit{threshold}(B, v_i)$ holds for all i such that $1 \leq i \leq k$. Then u belongs to $\mathit{losers}(B_0)$.

Proof. Suppose not. Since z does not belong to $\mathit{priorities}(B)$, we deduce that u belongs to $\mathit{winners}(B_0)$, and hence that z belongs to $\mathit{priorities}(B_0)$.

For i ranging from 1 to k, let B_i denote the IUAP $B_{i-1} + u_i$ where $u_i = (\alpha_i, \{(v_i, x_i)\}, z_i)$ and z_i is a real number satisfying the following conditions: z_i does not belong to $\mathit{priorities}(B_{i-1})$; $z < z_i$; $(x_i, z_i) < \mathit{threshold}(B, v_i)$. Since z_i does not belong to $\mathit{priorities}(B_{i-1})$, we deduce that u_i belongs to either $\mathit{winners}(B_i)$ or $\mathit{losers}(B_i)$ for $1 \leq i \leq k$. Then, by Lemmas 19 and 20, we deduce that u_i belongs to $\mathit{losers}(B_i)$ for $1 \leq i \leq k$. By repeated application of Lemma 17, we find that $\mathit{priorities}(B_i) = \mathit{priorities}(B_0)$ for $1 \leq i \leq k$, and hence that z belongs to $\mathit{priorities}(B_k)$.

We claim that u belongs to $\mathit{winners}(B_k)$. To prove this claim, let t denote the unique multibidder in B_k for which $\mathit{priority}(t) = \mathit{priority}(u)$. Let ℓ denote $|\mathit{bidders}(t)|$, and observe that $u = \mathit{bidder}(t, \ell)$. Furthermore, since z does not belong to $\mathit{priorities}(B)$, we deduce that $\mathit{bidder}(t, i)$ belongs to $\mathit{losers}(B)$ for $1 \leq i < \ell$. By repeated application of Lemma 13, we deduce that $\mathit{bidder}(t, i)$ belongs to $\mathit{losers}(B_k)$ for $1 \leq i < \ell$. Since z belongs to $\mathit{priorities}(B_k)$, the claim follows.

Let M denote a greedy MWMCM of $\mathit{uap}(B_k)$. Since u belongs to $\mathit{winners}(B_k)$, there is a unique integer i, $1 \leq i \leq k$, such that M contains edge (u, v_i). Let i denote this integer. Since z_i does not belong to $\mathit{priorities}(B_k)$, we know that u_i belongs to $\mathit{losers}(B_k)$ and hence that u_i is not matched in any greedy MWMCM of $\mathit{uap}(B_k)$. By Lemma 3, we deduce that $(x_i, z_i) < (x_i, z)$. Hence $z_i < z$, contradicting the definition of z_i. □

Lemma 24. Let $B_0 = (T, V)$ be an IUAP, let z be a real that is not equal to the priority of any multibidder in T, let k be a nonnegative integer, and for i ranging from 1 to k, let B_i denote the IUAP $B_{i-1} + u_i$, where $\mathit{priority}(u_i) = z$. Let I denote the set of all integers i in $\{1, \ldots, k\}$ such that there exists an item v in V for which $(w(u_i, v), z) > \mathit{threshold}(B_0, v)$. If I is empty, then z does not belong to $\mathit{priorities}(B_k)$. Otherwise, u_j belongs to $\mathit{winners}(B_k)$, where j denotes the minimum integer in I.

Proof. If I is empty, then by repeated application of Lemmas 21 and 23, we find that u_i belongs to $losers(B_i)$ for $1 \leq i \leq k$. By repeated application of Lemma 13, we deduce that u_i belongs to $losers(B_k)$ for $1 \leq i \leq k$. It follows that z does not belong to $priorities(B_k)$, as required.

Now assume that I is nonempty, and let j denote the minimum integer in I. Arguing as in the preceding paragraph, we find that z does not belong to $priorities(B_{j-1})$. By repeated application of Lemma 21, we deduce that $threshold(B_{j-1}, v) = threshold(B_0, v)$ for all items v in V. Thus Lemma 22 implies that u_j belongs to $winners(B_j)$. Then, since $u_{j+1}, \ldots, u_k$ all have the same priority as u_j, it is easy to argue by Lemma 10 that $uap(B_k) = uap(B_j)$, and hence u_j belongs to $winners(B_k)$, as required. □

Proof of Lemma 15. It is easy to see that the claims of the lemma follow from Lemma 24. □

4 Stable Marriage with Indifferences

The *stable marriage model with incomplete and weak preferences (SMIW)* involves a set P of men and a set Q of women. The preference relation of each man p in P is specified as a binary relation $\succeq_p$ over $Q \cup \{\varnothing\}$ that satisfies transitivity and totality, where $\varnothing$ denotes being unmatched. Similarly, the preference relation of each woman q in Q is specified as a binary relation $\succeq_q$ over $P \cup \{\varnothing\}$ that satisfies transitivity and totality, where $\varnothing$ denotes being unmatched. To allow indifferences, the preference relation is not required to satisfy anti-symmetry. We will use $\succ_p$ and $\succ_q$ to denote the asymmetric part of $\succeq_p$ and $\succeq_q$ respectively.

A matching is a function μ from P to $Q \cup \{\varnothing\}$ such that for any woman q in Q, there exists at most one man p in P for which $\mu(p) = q$. Given a matching μ and a woman q in Q, we denote

$$\mu(q) = \begin{cases} p & \text{if } \mu(p) = q \\ \varnothing & \text{if there is no man } p \text{ in } P \text{ such that } \mu(p) = q \end{cases}$$

A matching μ is *individually rational* if for any man p in P and woman q in Q such that $\mu(p) = q$, we have $q \succeq_p \varnothing$ and $p \succeq_q \varnothing$. A pair (p, q') in $P \times Q$ is said to form a *strongly blocking pair* for a matching μ if $q' \succ_p \mu(p)$ and $p \succ_{q'} \mu(q')$. A matching is *weakly stable* if it is individually rational and does not admit a strongly blocking pair.

For any matching μ and μ', we say that the binary relation $\mu \succeq \mu'$ holds if for every man p in P and woman q in Q, we have $\mu(p) \succeq_p \mu'(p)$ and $\mu(q) \succeq_q \mu'(q)$. Let $\succ$ be the asymmetric part of $\succeq$. We say that a matching μ *Pareto-dominates* another matching μ' if $\mu \succ \mu'$. We say that a matching is *Pareto-optimal* if it is not Pareto-dominated by any other matching. A matching is *Pareto-stable* if it is Pareto-optimal and weakly stable.

A *mechanism* is an algorithm that, given $(P, Q, (\succeq_p)_{p \in P}, (\succeq_q)_{q \in Q})$, produces a matching μ. A mechanism is said to be *strategyproof (for the men)* if for any man p in P expressing preference $\succeq'_p$ instead of his true preference $\succeq_p$, we have $\mu(p) \succeq_p \mu'(p)$, where μ and μ' are the matchings produced by the mechanism given $\succeq_p$ and $\succeq'_p$, respectively, when all other inputs are fixed.

Without loss of generality, we may assume that the number of men is equal to the number of women. So, $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$.

4.1 Algorithm

Algorithm MATCHSMIW:

1. For each $1 \leq j \leq n$,
   (a) Let p_{j-2n} denote $\varnothing$.
   (b) Convert the preference relation $\succeq_{q_j}$ of woman q_j into utility function $\psi_{q_j} \colon P \cup \{\varnothing\} \to \mathbb{R}$, such that for any i and i' in $\{1, \ldots, n\} \cup \{j - 2n\}$, we have $p_i \succeq_{q_j} p_{i'}$ if and only if $\psi_{q_j}(p_i) \geq \psi_{q_j}(p_{i'})$. This utility assignment should not depend on the preferences of the men.
   (c) Construct an item v_j corresponding to woman q_j.
   (d) Construct a reserve multibidder $t_{j-2n} = (\sigma_{j-2n}, z_{j-2n})$ for item v_j with priority $z_{j-2n} = j - 2n$, such that $\mathit{items}(\sigma_{j-2n}(1)) = \{v_j\}$ and $w(\sigma_{j-2n}(1), v_j) = \psi_{q_j}(p_{j-2n})$.
2. For each $n < j \leq 2n$,
   (a) Let q_j denote $\varnothing$.
   (b) Construct a dummy item v_j corresponding to q_j.
   (c) Construct a reserve multibidder $t_{j-2n} = (\sigma_{j-2n}, z_{j-2n})$ for dummy item v_j with priority $z_{j-2n} = j - 2n$, such that $\mathit{items}(\sigma_{j-2n}(1)) = \{v_j\}$ and $w(\sigma_{j-2n}(1), v_j) = 0$.
3. For each $1 \leq i \leq n$,
   (a) Partition the set $\{1, \ldots, n\} \cup \{n + i\}$ of woman indices into tiers $\tau_i(1), \ldots, \tau_i(K_i)$ according to the preference relation of man p_i, such that for any j in $\tau_i(k)$ and j' in $\tau_i(k')$, we have $q_j \succeq_{p_i} q_{j'}$ if and only if $k \leq k'$.
   (b) For j in $\{1, \ldots, n\} \cup \{n + i\}$, denote tier number $\kappa_i(q_j)$ as the unique k such that j in $\tau_i(k)$.
   (c) Construct a multibidder $t_i = (\sigma_i, z_i)$ with priority $z_i = i$ corresponding to man p_i. The multibidder t_i has K_i bidders. For each bidder $\sigma_i(k)$ we define $\mathit{items}(\sigma_i(k))$ as $\{v_j \mid j \in \tau_i(k)\}$ and $w(\sigma_i(k), q_j)$ as $\psi_{q_j}(p_i)$, where $\psi_{q_{n+i}}(p_i)$ is defined to be 0.
4. $(T, V) = (\{t_i \mid -2n < i \leq n\}, \{v_j \mid 1 \leq j \leq 2n\})$.
5. $(U, V) = \mathit{uap}(T, V)$.
6. Compute a greedy MWMCM M_0 of UAP (U, V) as described in Sect. 2.1.
7. Output matching μ such that for all $1 \leq i \leq n$ and $1 \leq j \leq 2n$, we have $\mu(p_i) = q_j$ if and only if $\sigma_i(k)$ is matched to item v_j in M_0 for some k.

4.2 Correctness

Lemma 25. Algorithm MATCHSMIW produces a valid matching.

Proof. First, we show that for any man p_i where $1 \leq i \leq n$, there exists at most one j in $\{1, \ldots, 2n\}$ such that bidder $\sigma_i(k)$ is matched to item v_j in M_0 for some k. For the sake of contradiction, suppose bidder $\sigma_i(k)$ is matched to item v_j and bidder $\sigma_i(k')$ is matched to item $v_{j'}$ in M_0 for some k and k' where $j \neq j'$. By Lemma 12, we have $k = k'$. Therefore, bidder $\sigma_i(k) = \sigma_i(k')$ is matched in M_0 to both v_j and $v_{j'}$, which is a contradiction.

Next, we show that for any man p_i where $1 \le i \le n$, there exists at least one j in $\{1, \ldots, 2n\}$ such that bidder $\sigma_i(k)$ is matched to item v_j in M_0 for some k. For the sake of contradiction, suppose bidder $\sigma_i(k)$ is unmatched in M_0 for all k. Let j denote $n + i$ and let k denote $\kappa_i(q_j)$. By Lemma 12, the set U contains bidder $\sigma_i(k)$. Since $\sigma_i(k)$ is not matched to dummy item v_j in M_0, the dummy item v_j is matched to its reserve bidder $\sigma_{j-2n}(1)$ in M_0. So, by Lemma 11, we have

$$(0, z_i) = (w(\sigma_i(k), v_j), z_i) < (w(\sigma_{j-2n}(1), v_j), z_{j-2n}) = (0, z_{j-2n}),$$

which contradicts $z_i = i > j - 2n = z_{j-2n}$.

This shows that $\mu(p_i)$ is well-defined for all men p_i where $1 \le i \le n$. Furthermore, since each item v_j where $1 \le j \le n$ is matched to at most one bidder in M_0, each woman q_j is matched to at most one man p_i in μ where $1 \le i \le n$. Hence, μ is a valid matching. $\square$

Lemma 26. Algorithm MATCHSMIW produces an individually rational matching.

Proof. We have shown in Lemma 25 that μ is a valid matching. Consider man p_i and woman q_j such that $\mu(p_i) = q_j$, where i and j belong to $\{1, \ldots, n\}$. Let k denote $\kappa_i(q_j)$ and let k' denote $\kappa_i(q_{n+i})$. It suffices to show that $k \le k'$ and $\psi_{q_j}(p_i) \ge \psi_{q_j}(p_{j-2n})$.

For the sake of contradiction, suppose $k > k'$. Since bidder $\sigma_i(k)$ is matched to item v_j in M_0, by Lemma 12 the set U contains bidder $\sigma_i(k')$. Since bidder $\sigma_i(k')$ is not matched to dummy item v_{n+i} in M_0, the dummy item v_{n+i} is matched to its reserve bidder $\sigma_{i-n}(1)$ in M_0. So, by Lemma 11, we have

$$(0, z_i) = (w(\sigma_i(k'), v_{n+i}), z_i) < (w(\sigma_{i-n}(1), v_{n+i}), z_{i-n}) = (0, z_{i-n}),$$

which contradicts $z_i = i > i - n = z_{i-n}$.

For the sake of contradiction, suppose $\psi_{q_j}(p_i) < \psi_{q_j}(p_{j-2n})$. Since bidder $\sigma_i(k)$ is matched to item v_j in M_0, the reserve bidder $\sigma_{j-2n}(1)$ is unmatched in M_0. So, by Lemma 11, we have

$$(\psi_{q_j}(p_{j-2n}), z_{j-2n}) = (w(\sigma_{j-2n}(1), v_j), z_{j-2n}) < (w(\sigma_i(k), v_j), z_i) = (\psi_{q_j}(p_i), z_i),$$

which contradicts the assumption that $\psi_{q_j}(p_i) < \psi_{q_j}(p_{j-2n})$. $\square$

Lemma 27. Algorithm MATCHSMIW produces a weakly stable matching.

Proof. By Lemma 26, it remains only to show that μ does not admit a strongly blocking pair. Consider man p_i and woman $q_{j'}$, where i and j belong to $\{1, \ldots, n\}$. We want to show that $(p_i, q_{j'})$ does not form a strongly blocking pair. Let q_j denote $\mu(p_i)$ and let $p_{i'}$ denote $\mu(q_{j'})$, where j belongs to $\{1, \ldots, n\} \cup \{n + i\}$ and i' belongs to $\{1, \ldots, n\} \cup \{j' - 2n\}$. It suffices to show that either $\kappa_i(q_j) \le \kappa_i(q_{j'})$ or $\psi_{q_{j'}}(p_{i'}) \ge \psi_{q_{j'}}(p_i)$. For the sake of contradiction, suppose $\kappa_i(q_j) > \kappa_i(q_{j'})$ and $\psi_{q_{j'}}(p_{j'}) < \psi_{q_{j'}}(p_i)$.

Let k denote $\kappa_i(q_j)$ and let k' denote $\kappa_i(q_{j'})$. Since $p_{i'} = \mu(q_{j'})$, there exists k'' such that bidder $\sigma_{i'}(k'')$ is matched to $v_{j'}$ in M_0. Since $\sigma_i(k)$ is matched in M_0 and $k' < k$, by Lemma 12 the set U contains bidder $\sigma_i(k')$. Since $\sigma_i(k')$ is unmatched in M_0, by Lemma 11 we have

$$(\psi_{q_{j'}}(p_i), i) = (w(\sigma_i(k'), v_{j'}), z_i) < (w(\sigma_{i'}(k''), v_{j'}), z_{i'}) = (\psi_{q_{j'}}(p_{i'}), i'),$$

which contradicts the assumption that $\psi_{q_{j'}}(p_i) > \psi_{q_{j'}}(p_{j'})$. $\square$

Lemma 28. Let μ be the matching produced by the algorithm MATCHSMIW and let μ' be a matching such that $\mu'(p) \succeq_p \mu(p)$ for every man p in P and

$$\sum_{q \in Q} \psi_q(\mu'(q)) \geq \sum_{q \in Q} \psi_q(\mu(q)).$$

Then, we have $\mu(p) \succeq_p \mu'(p)$ for every man p in P and

$$\sum_{q \in Q} \psi_q(\mu'(q)) = \sum_{q \in Q} \psi_q(\mu(q)).$$

Proof. For any i such that $1 \leq i \leq n$, let k_i denote $\kappa_i(\mu(p_i))$ and let k'_i denote $\kappa_i(\mu'(p_i))$. For any i such that $-2n < i \leq 0$, let k_i and k'_i denote 1.

Below we use μ' to construct an MWMCM M'_0 of (U, V). We give the construction of M'_0 first, and then argue that M'_0 is an MWMCM of (U, V). Let M'_0 denote the set of bidder-item pairs $(\sigma_i(k'_i), v_j)$ such that one of the following conditions holds: $1 \leq i \leq n$ and $1 \leq j \leq n$ and $\mu'(p_i) = q_j$; $1 \leq i \leq n$ and $j = n + i$ and $\mu'(p_i) = \varnothing$; $-2n < i \leq n$ and $j = i + 2n$ and $\mu'(q_j) = \varnothing$; $-n < i \leq 0$ and $j = i + 2n$ and $\mu'(p_{i+n}) \neq \varnothing$. It is easy to see that M'_0 is a valid matching that matches all of the items in V. Notice that for any $1 \leq i \leq n$, since $\mu'(p_i) \succeq_{p_i} \mu(p_i)$, we have $k'_i \leq k_i$. So, by Lemma 12, the set U contains all bidders $\sigma_i(k'_i)$. Hence, M'_0 is an MCM of (U, V). Furthermore, it is easy to see that

$$w(M'_0) = \sum_{1 \leq j \leq n} \psi_{q_j}(\mu'(q_j)) \geq \sum_{1 \leq j \leq n} \psi_{q_j}(\mu(q_j)) = w(M_0).$$

Thus M'_0 is an MWMCM of (U, V), and we have

$$\sum_{1 \leq j \leq n} \psi_{q_j}(\mu'(q_j)) = \sum_{1 \leq j \leq n} \psi_{q_j}(\mu(q_j)).$$

To show that $\mu(p_i) \succeq_{p_i} \mu'(p_i)$ for all $1 \leq i \leq n$, it suffices to show that $k_i = k'_i$ for all $1 \leq i \leq n$. For the sake of contradiction, suppose there exists a maximum i' such that $k_{i'} > k'_{i'}$. Notice that bidder $\sigma_{i'}(k'_{i'})$ is unmatched in every greedy MWMCM of (U, V), for otherwise $k_{i'} \leq k'_{i'}$ by Lemma 12. Since bidder $\sigma_{i'}(k'_{i'})$ is matched in M'_0, by Lemma 4 there exists a bidder $\sigma_i(k_i)$ with priority $z_i > z_{i'}$ such that $\sigma_i(k_i)$ is matched in M_0 but unmatched in M'_0. Since $i = z_i > z_{i'} = i'$, by the maximality of i' we have $k_i = k'_i$. Thus bidder $\sigma_i(k'_i) = \sigma_i(k_i)$ is unmatched in M'_0, a contradiction. $\square$

Lemma 29. Let μ be the matching produced by the algorithm MATCHSMIW and μ' be a matching such that $\mu' \succeq \mu$. Then, $\mu \succeq \mu'$.

Proof. Since $\mu' \succeq \mu$, we have $\mu'(p_i) \succeq_{p_i} \mu(p_i)$ and $\psi_{q_j}(\mu'(q_j)) \geq \psi_{q_j}(\mu(q_j))$ for every i and j in $\{1, \ldots, n\}$. So, by Lemma 28, we have $\mu(p_i) \succeq_{p_i} \mu'(p_i)$ for every i in $\{1, \ldots, n\}$ and

$$\sum_{1 \leq j \leq n} \psi_{q_j}(\mu'(q_j)) = \sum_{1 \leq j \leq n} \psi_{q_j}(\mu(q_j)).$$

Therefore, $\psi_{q_j}(\mu'(q_j)) = \psi_{q_j}(\mu(q_j))$ for every j in $\{1, \ldots, n\}$. This shows that $\mu \succeq \mu'$. $\square$

Lemma 30. Consider the algorithm MATCHSMIW. Suppose $\mu(p_i) = q_j$, where $1 \le i \le n$ and j belongs to $\{1, \ldots, n\} \cup \{n+i\}$. Then, we have

$$(\psi_{q_j}(p_i), i) \ge \mathit{threshold}((T - t_i, V), v_j). \tag{3}$$

Furthermore, for all j' in $\{1, \ldots, n\} \cup \{n+i\}$ such that $\kappa_i(q_{j'}) < \kappa_i(q_j)$, we have

$$(\psi_{q_{j'}}(p_i), i) < \mathit{threshold}((T - t_i, V), v_{j'}). \tag{4}$$

Proof. Let k denote $\kappa_i(q_j)$. Since $\mu(p_i) = q_j$, we know that bidder $\sigma_i(k)$ is matched to item v_j in M_0. So, inequality (1) in Lemma 15 implies inequality (3), because $w(\sigma_i(k), v_j) = \psi_{q_j}(p_i)$ and $z_i = i$.

Now, suppose $\kappa_i(q_{j'}) < \kappa_i(q_j)$. Let k' denote $\kappa_i(q_{j'})$. Since $k' < k$, inequality (2) in Lemma 15 implies inequality (4), because $w(\sigma_i(k'), v_{j'}) = \psi_{q_{j'}}(p_i)$ and $z_i = i$. □

Theorem 1. The algorithm MATCHSMIW is a strategyproof Pareto-stable mechanism for the stable marriage problem with incomplete and weak preferences (for any fixed choice of utility assignment).

Proof. We have shown in Lemma 27 that the algorithm produces a weakly stable matching. Moreover, Lemma 29 shows that the weakly stable matching produced is not Pareto-dominated by any other matching. Hence, the algorithm produces a Pareto-stable matching. It remains to show that the algorithm is a strategyproof mechanism.

Suppose man p_i expresses $\succeq'_{p_i}$ instead of his true preference relation $\succeq_{p_i}$, where $1 \le i \le n$. Let μ and μ' be the resulting matchings given $\succeq_{p_i}$ and $\succeq'_{p_i}$ respectively. Let q_j denote $\mu(p_i)$ and let $q_{j'}$ denote $\mu'(p_i)$, where j and j' belong to $\{1, \ldots, n\} \cup \{n+i\}$. Let k denote $\kappa_i(q_j)$ and let k' denote $\kappa_i(q_{j'})$, where $\kappa_i(\cdot)$ denotes the tier number with respect to $\succeq_{p_i}$. It suffices to show that $k \le k'$. For the sake of contradiction, suppose $k > k'$.

Let (T, V) be the iterated unit-demand auction, let t_i be the multibidder corresponding to man p_i, and let $v_{j'}$ be the item corresponding to woman $q_{j'}$ constructed in the algorithm given input $\succeq_{p_i}$. Since $\mu(p_i) = q_j$, by equation (4) of Lemma 30, we have

$$(\psi_{q_{j'}}(p_i), i) < \mathit{threshold}((T - t_i, V), v_{j'}).$$

Now, consider the behavior of the algorithm when preference relation $\succeq_{p_i}$ is replaced with $\succeq'_{p_i}$. Let (T', V') be the IUAP, let t'_i be the multibidder corresponding to man p_i, and let $v'_{j'}$ be the item corresponding to woman $q_{j'}$ constructed in the algorithm given input $\succeq'_{p_i}$. Since $\mu'(p_i) = q_{j'}$, by equation (3) of Lemma 30, we have

$$(\psi_{q_{j'}}(p_i), i) \ge \mathit{threshold}((T' - t'_i, V'), v'_{j'}).$$

Notice that in the algorithm MATCHSMIW, the only part of the IUAP instance that depends on the preference of man p_i is the multibidder corresponding to man p_i. In particular, we have $T - t_i = T' - t'_i$, $V = V'$, and $v_{j'} = v'_{j'}$. Hence, we get

$$\begin{aligned}(\psi_{q_{j'}}(p_i), i) &< \mathit{threshold}((T - t_i, V), v_{j'}) \\ &= \mathit{threshold}((T' - t'_i, V'), v'_{j'}) \\ &\le (\psi_{q_{j'}}(p_i), i),\end{aligned}$$

which is a contradiction. □

5 College Admissions with Indifferences

The *college admissions model with weak preferences (CAW)* involves a set P of students and a set Q of colleges. The preference relation of each student p in P is specified as a binary relation $\succeq_p$ over $Q \cup \{\varnothing\}$ that satisfies transitivity and totality, where $\varnothing$ denotes being unmatched. The preference relation of each college q in Q over individual students is specified as a binary relation $\succeq_q$ over $P \cup \{\varnothing\}$ that satisfies transitivity and totality, where $\varnothing$ denotes being unmatched. Each college q in Q has an associated integer capacity $c_q > 0$. We will use $\succ_p$ and $\succ_q$ to denote the asymmetric parts of $\succeq_p$ and $\succeq_q$ respectively.

The colleges' preference relation over individual students can be extended to group preference using responsiveness. We say that a transitive and reflexive relation $\succeq'_q$ over the power set 2^P is *responsive to the preference relation* $\succeq_q$ if the following conditions hold: for any $S \subseteq P$ and p in $P \setminus S$, we have $p \succeq_q \varnothing$ if and only if $S \cup \{p\} \succeq'_q S$; for any $S \subseteq P$ and any p and p' in $P \setminus S$, we have $p \succeq p'$ if and only if $S \cup \{p\} \succeq'_q S \cup \{p'\}$. Furthermore, we say that a relation $\succeq'_q$ is *minimally responsive to the preference relation* $\succeq_q$ if it is responsive to the preference relation $\succeq_q$ and does not strictly contain another relation that is responsive to the preference relation $\succeq_q$.

A *(capacitated) matching* is a function μ from P to $Q \cup \{\varnothing\}$ such that for any college q in Q, there exists at most c_q students p in P for which $\mu(p) = q$. Given a matching μ and a college q in Q, we let $\mu(q)$ denote $\{p \in P \mid \mu(p) = q\}$.

A matching μ is *individually rational* if for any student p in P and college q in Q such that $\mu(p) = q$, we have $q \succeq_p \varnothing$ and $p \succeq_q \varnothing$. A pair (p', q) in $P \times Q$ is said to form a *strongly blocking pair* for a matching μ if $q \succ_{p'} \mu(p')$ and at least one of the following two conditions holds: (1) there exists a student p in P such that $\mu(p) = q$ and $p' \succ_q p$; (2) $|\mu(q)| < c_q$ and $p' \succ_q \varnothing$. A matching is *weakly stable* if it is individually rational and does not admit a strongly blocking pair.

Let $\succeq'_q$ be the group preference associated with college q in Q. For any matching μ and μ', we say that the binary relation $\mu \succeq \mu'$ holds if for every student p in P and college q in Q, we have $\mu(p) \succeq_p \mu'(p)$ and $\mu(q) \succeq'_q \mu'(q)$. Let $\succ$ be the asymmetric part of $\succeq$. We say that a matching μ *Pareto-dominates* another matching μ' if $\mu \succ \mu'$. We say that a matching is *Pareto-optimal* if it is not Pareto-dominated by any other matching. A matching is *Pareto-stable* if it is Pareto-optimal and weakly stable.

A *mechanism* is an algorithm that, given $(P, Q, (\succeq_p)_{p \in P}, (\succeq_q)_{q \in Q}, (c_q)_{q \in Q})$, produces a matching μ. A mechanism is said to be *strategyproof (for the students)* if for any student p in P expressing preference $\succeq'_p$ instead of their true preference $\succeq_p$, we have $\mu(p) \succeq_p \mu'(p)$, where μ and μ' are the matchings produced by the mechanism given $\succeq_p$ and $\succeq'_p$, respectively, when all other inputs are fixed.

Without loss of generality, we may assume that the number of students equals the total capacities of all the colleges. So, $P = \{p_i\}_{1 \le i \le |P|}$ and $Q = \{q_j\}_{1 \le j \le |Q|}$ such that $|P| = \sum_{1 \le j \le |Q|} c_{q_j}$.

5.1 Algorithm

Algorithm MATCHCAW:

1. For each $1 \le i \le |P|$, construct man p'_i corresponding to student p_i.
2. For each $1 \le j \le |Q|$, construct women $q'_{j1}, \ldots, q'_{jc}$ corresponding to college q_j with capacity $c = c_{q_j}$.

3. $(P', Q') = (\{p'_i \mid 1 \le i \le |P|\}, \{q'_{jk} \mid 1 \le j \le |Q| \text{ and } 1 \le k \le c_{q_j}\})$.
4. Let p_0 denote $\varnothing$. Let p'_0 denote $\varnothing$.
5. Let q_0 denote $\varnothing$. Let q'_{00} denote $\varnothing$.
6. For each $1 \le i \le |P|$, define the preference relation $\succeq_{p'_i}$ over $Q' \cup \{q'_{00}\}$ for man p'_i using the preference relation of his corresponding student, such that $q'_{jk} \succeq_{p'_i} q'_{j'k'}$ if and only if $q_j \succeq_{p_i} q_{j'}$.
7. For each $1 \le j \le |Q|$ and $1 \le k \le c_{q_j}$, define the preference relation $\succeq_{q'_{jk}}$ over $P' \cup \{p'_0\}$ for woman q'_{jk} using the preference relation of her corresponding college, such that $p'_i \succeq_{q'_{jk}} p'_{i'}$ if and only if $p_i \succeq_{q_j} p_{i'}$.
8. Compute SMIW matching $\mu_0 = \text{MATCHSMIW}(P', Q', (\succeq_{p'})_{p' \in P'}, (\succeq_{q'})_{q' \in Q'})$.
9. Output matching μ, such that for all $1 \le i \le |P|$ and $0 \le j \le |Q|$, we have $\mu(p_i) = q_j$ if and only if $\mu_0(p'_i) = q'_{jk}$ for some k.

5.2 Correctness

Lemma 31. Algorithm MATCHCAW produces an individually rational matching.

Proof. It is easy to see that μ satisfies the capacity constraints because each college q_j is associated with c_{q_j} women q'_{jk} and each woman can be matched with at most one man in μ_0 by Lemma 25.

The individual rationality of μ follows from the individual rationality of μ_0. Let p_i in P and q_j in Q such that $\mu(p_i) = q_j$. Then $\mu_0(p'_i) = q'_{jk}$ for some k. By Lemma 26, we have $q'_{jk} \succeq_{p'_i} \varnothing$ and $p'_i \succeq_{q'_{jk}} \varnothing$. Hence, $q_j \succeq_{p_i} \varnothing$ and $p_i \succeq_{q_j} \varnothing$. □

Lemma 32. Algorithm MATCHCAW produces a weakly stable matching.

Proof. By Lemma 31, it remains only to show that μ does not admit a strongly blocking pair. Consider student $p_{i'}$ in P and college q_j in Q. We want to show that $(p_{i'}, q_j)$ does not form a strongly blocking pair using the weak stability of μ_0.

Let $q'_{j'k'}$ denote $\mu_0(p'_{i'})$. It is possible that $q'_{j'k'} = \varnothing$, in which case $j' = k' = 0$. For $1 \le k \le c_{q_j}$, let p'_{i_k} denote $\mu_0(q'_{jk})$, where p'_{i_k} belongs to $P' \cup \{p'_0\}$. By Lemma 27, for any $1 \le k \le c_{q_j}$, either $q'_{j'k'} \succeq_{p'_{i'}} q'_{jk}$ or $p'_{i_k} \succeq_{q'_{jk}} p'_{i'}$, for otherwise $(p'_{i'}, q'_{jk})$ forms a strongly blocking pair.

Suppose $q'_{j'k'} \succeq_{p'_{i'}} q'_{jk}$ for some $1 \le k \le c_{q_j}$. Then, we have $q_{j'} \succeq_{p_{i'}} q_j$. So, $(p_{i'}, q_j)$ does not form a strongly blocking pair.

Otherwise, $p'_{i_k} \succeq_{q'_{jk}} p'_{i'}$ for all $1 \le k \le c_{q_j}$. Then, we have $p_{i_k} \succeq_{q_j} p_{i'}$ for all $1 \le k \le c_{q_j}$. In particular, we have $p_{i_k} \succeq_{q_j} p_{i'}$ for all students p_{i_k} in P such that $\mu(p_{i_k}) = q_j$. Furthermore, if $|\mu(q_j)| < c_{q_j}$, then $p_{i_k} = \varnothing$ for some $1 \le k \le c_{q_j}$. Hence, $\varnothing \succeq_{q_j} p_{i'}$. This shows that $(p_{i'}, q_j)$ does not form a strongly blocking pair. □

Lemma 33. Suppose that for every college q in Q, the group preference relation $\succeq'_q$ is minimally responsive to $\succeq_q$. Let μ be the matching produced by the algorithm MATCHCAW and let μ' be a matching such that $\mu' \succeq \mu$. Then $\mu \succeq \mu'$.

Proof. Since μ' is a matching that satisfies the capacity contraints, we can construct an SMIW matching $\mu'_0 \colon P' \to Q' \cup \{q'_{00}\}$ such that for all $1 \le i \le |P|$ and $0 \le j \le |Q|$, we have $\mu'(p_i) = q_j$ if and only if $\mu_0(p'_i) = q'_{jk}$ for some k.

Since $\mu' \succeq \mu$, we have $\mu'(p_i) \succeq_{p_i} \mu(p_i)$ for every $1 \leq i \leq |P|$ and $\mu'(q_j) \succeq'_{q_j} \mu(q_j)$ for every $1 \leq j \leq |Q|$. So, $\mu'_0(p'_i) \succeq_{p'_i} \mu_0(p'_i)$ for every $1 \leq i \leq |P|$ and

$$\sum_{1 \leq k \leq c_{q_j}} \psi_{q'_{jk}}(\mu'_0(q'_{jk})) \geq \sum_{1 \leq k \leq c_{q_j}} \psi_{q'_{jk}}(\mu_0(q'_{jk}))$$

for every $1 \leq j \leq |Q|$. Hence, by Lemma 28, we have $\mu_0(p'_i) \succeq_{p'_i} \mu'_0(p'_i)$ for every $1 \leq i \leq |P|$ and

$$\sum_{1 \leq j \leq |Q|} \sum_{1 \leq k \leq c_{q_j}} \psi_{q'_{jk}}(\mu'_0(q'_{jk})) = \sum_{1 \leq j \leq |Q|} \sum_{1 \leq k \leq c_{q_j}} \psi_{q'_{jk}}(\mu_0(q'_{jk})).$$

Therefore, we have $\mu(p_i) \succeq_{p_i} \mu'(p_i)$ for every $1 \leq i \leq |P|$ and

$$\sum_{1 \leq k \leq c_{q_j}} \psi_{q'_{jk}}(\mu'_0(q'_{jk})) = \sum_{1 \leq k \leq c_{q_j}} \psi_{q'_{jk}}(\mu_0(q'_{jk}))$$

for every $1 \leq j \leq |Q|$. This shows that $\mu(q_j) \succeq'_{q_j} \mu'(q_j)$ for every $1 \leq j \leq |Q|$. Thus, $\mu \succeq \mu'$. □

Theorem 2. Suppose that for every college q in Q, the group preference relation $\succeq'_q$ is minimally responsive to $\succeq_q$. The algorithm MATCHCAW is a strategyproof Pareto-stable mechanism for the college admissions problem with weak preferences (for any fixed choice of utility assignment).

Proof. We have shown in Lemma 32 that the algorithm produces a weakly stable matching. Moreover, Lemma 33 shows that the weakly stable matching produced is not Pareto-dominated by any other matching. Hence, the algorithm produces a Pareto-stable matching.

To show that the algorithm provides a strategyproof mechanism, suppose student p_i expresses $\succeq'_{p_i}$ instead of their true preference relation $\succeq_{p_i}$, where $1 \leq i \leq |P|$. Let μ and μ' be the resulting matchings given $\succeq_{p_i}$ and $\succeq'_{p_i}$, respectively. Let μ_0 and μ'_0 be the SMIW matching produced by the algorithm given $\succeq_{p_i}$ and $\succeq'_{p_i}$, respectively.

Notice that in the algorithm MATCHCAW, the only part of the stable marriage instance that depends on the preferences of student p_i is the preference relation corresponding to man p'_i. Since algorithm MATCHSMIW is strategyproof by Theorem 1, we have $\mu_0(p'_i) \succeq_{p'_i} \mu'_0(p'_i)$ where $\succeq_{p'_i}$ is the preference relation of man p'_i in the algorithm given $\succeq_{p_i}$. Hence, $\mu(p_i) \succeq_{p_i} \mu'(p_i)$. □

We remark that algorithm MATCHCAW admits an $O(n^4)$-time implementation, where n is the sum of the number of students and the total capacities of all the colleges, because the reduction from CAW to IUAP takes $O(n^2)$ time, and steps 5 and 6 of MATCHSMIW algorithm can be implemented in $O(n^4)$ time using the version of the incremental Hungarian method discussed in Sections 2.1 and 3.1.

5.3 Further Discussion

In our SMIW and CAW algorithms, we transform the preference relations of the women and colleges into real-valued utility functions. One way to do this is to set $\psi_q(p)$ to the number of p' in $P \cup \{\varnothing\}$ such that $p \succeq_q p'$. This is by no means the only way. In fact, different ways of assigning the utilities can affect the outcome. Nonetheless, our mechanisms remain strategyproof for the men as long as the utility assignment is fixed and independent of the preferences of the men, as shown in Theorems 1 and 2.

We can also consider the scenario where each college expresses their preferences directly in terms of a utility function instead of a preference relation. Such utility functions provide another way to extend preferences over individuals to group preferences. If a college q expresses the utility function ψ_q over individual students in $P \cup \{\varnothing\}$, we can define the *group preference induced by additive utility* ψ_q as a binary relation $\succeq'_q$ over 2^P such that $S \succeq'_q S'$ if and only if

$$\sum_{p \in S} (\psi_q(p) - \psi_q(\varnothing)) \geq \sum_{p \in S'} (\psi_q(p) - \psi_q(\varnothing)).$$

Then, our algorithm can accept the utility functions as input in lieu of constructing them by some utility assignment method. It is not hard to see that the mechanism remains Pareto-stable and strategyproof when the group preferences of the colleges are induced by additive utilities.

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A Modified Hungarian Step for UAPs

In this appendix, we establish Lemma 36, which enables us to modify the incremental Hungarian step as described in Sect. 2.1 so that we can efficiently find a greedy MWMCM of a UAP.

Let $A = (U, V)$ and $A' = A + u$ be UAPs, and let M be an MWMCM of A. We define $\mathit{digraph}(A, u, M)$ as the edge-weighted digraph that may be obtained by modifying the subgraph of A induced by the set of vertices $(\mathit{matched}(M) + u) \cup V$ as follows: for each edge that belongs to M, we direct the edge from item to bidder and leave the weight unchanged; for each edge that does not belong to M, we direct the edge from bidder to item and negate the weight.

Lemma 34. Let $A = (U, V)$ and $A' = A + u$ be UAPs, let M be an MWMCM of A, and let G denote $\mathit{digraph}(A, u, M)$. Then G does not contain any negative-weight cycles.

Proof. Such a cycle could not involve u (since u only has outgoing edges) so it has to be a negative-weight cycle that already existed before u was added, a contradiction since M is an MWMCM of A. $\square$

Let $A = (U, V)$ and $A' = A + u$ be UAPs, let M be an MWMCM of A, and let G denote $\mathit{digraph}(A, u, M)$. Then we define a nonempty set of bidders $\mathit{candidates}(A, u, M)$, and a nonempty set of directed paths $\mathit{paths}(A, u, M)$, as follows. Let U' denote the set of all bidders that are reachable from u via a directed path in G. Observe that U' is nonempty, since u belongs to U'. Furthermore, by Lemma 34, for each bidder u' in U', the weight of a shortest path in G from u to u' is well-defined. Let W denote the minimum, over all u' in U', of the weight of a shortest path in G from u to u'. Then we define $\mathit{candidates}(A, u, M)$ as the set of all u' in U' such that the weight of a shortest path in G from u to u' is equal to W, and we define $\mathit{paths}(A, u, M)$ as the set of all weight-W directed paths in G that start at u and terminate at some minimum-priority bidder in $\mathit{candidates}(A, u, M)$.

Let $A = (U, V)$ and $A' = A + u$ be UAPs, let M be a greedy MWMCM of A, and let P be a directed path in $\mathit{digraph}(A, u, M)$ from u to a bidder u'. (Note that P could be a path of length zero from u to u.) Let X denote the edges in M corresponding to item-to-bidder edges in P, and let Y denote the edges in A' corresponding to bidder-to-item edges in P. Since P alternates between item-to-bidder and bidder-to-item edges and terminates at a bidder, we deduce that $|X| = |Y|$ and that the set of edges $(M \setminus X) \cup Y$ is an MCM of A'. We define this MCM of A' as $\mathit{augment}(A, u, M, P)$.

Lemma 35. Let $A = (U, V)$ and $A' = A + u$ be UAPs, let M be a greedy MWMCM of A, and let M' denote a greedy MWMCM of A' minimizing $|M \oplus M'|$. Then there is a directed path P in $\mathit{paths}(A, u, M)$ such that $M' = \mathit{augment}(A, u, M, P)$.

Proof. Since every item in V is matched in both M and M', the edges of $M \oplus M'$ form a collection $\mathcal{S}$ of disjoint cycles and positive-length paths.

We begin by arguing that $\mathcal{S}$ does not contain any cycles. Suppose there is a cycle C in $\mathcal{S}$. Let X denote the edges of C that belong to $M \setminus M'$, and let Y denote the edges of C that belong to $M' \setminus M$. Since C is an even-length cycle, we have $|X| = |Y|$. Let M'' denote $(M \cup Y) \setminus X$, which is a matching in A since u is unmatched in M and hence does not belong to C. Since M is an MWMCM of A and $w(M'') = w(M) + w(Y) - w(X)$, we conclude that $w(X) \geq w(Y)$. Let M''' denote $(M' \cup X) \setminus Y$, which is a matching in A'. Since M' is an MWMCM of A' and

$w(M''') = w(M') + w(Y) - w(X)$, we conclude that $w(X) \leq w(Y)$. Thus $w(X) = w(Y)$ and hence $w(M''') = w(M')$, implying that M''' is an MWMCM of A'. Moreover, since M''' matches the same set of bidders as M', we find that M''' is a greedy MWMCM of A'. But this contradicts the definition of M' since $|M \oplus M'''| < |M \oplus M'|$.

Next we argue that if Q is a positive-length path in $\mathcal{S}$, then u is an endpoint of Q. Suppose there is a path Q in $\mathcal{S}$ such that u is not an endpoint of Q. Thus u does not appear on Q since u is unmatched in M. Let X denote the edges of Q that belong to $M \setminus M'$, and let Y denote the edges of Q that belong to $M' \setminus M$. Let M'' denote $(M \cup Y) \setminus X$, which is a matching in A since u does not belong to Q. Since M is an MWMCM of A and $w(M'') = w(M) + w(Y) - w(X)$, we conclude that $|X| \geq |Y|$ and $w(X) \geq w(Y)$. Let M''' denote $(M' \cup X) \setminus Y$, which is a matching in A'. Since M' is an MWMCM of A' and $w(M''') = w(M') + w(Y) - w(X)$, we conclude that $|X| \leq |Y|$ and $w(X) \leq w(Y)$. Thus $|X| = |Y|$ and $w(X) = w(Y)$, and hence $w(M'') = w(M)$ and $w(M''') = w(M')$, implying that M'' is an MWMCM of A and M''' is an MWMCM of A'. Since $|X| = |Y|$, Q is an even-length path. Since every item in V is matched in both M and M', both endpoints of Q are bidders. Since Q has positive length, one endpoint, call it u_0, is matched in M but not in M', and the other endpoint, call it u_1, is matched in M' but not in M. Since M is a greedy MWMCM of A and M'' is an MWMCM of A, we deduce that the priority of u_0 is at least that of u_1. Since M' is a greedy MWMCM of A' and M''' is an MWMCM of A', we deduce that the priority of u_0 is at most that of u_1. Thus the priority of u_0 is equal to that of u_1. It follows that the sum of the priorities of the bidders matched by M''' is equal to the sum of the priorities of the bidders matched by M'. Hence M''' is a greedy MWMCM of A'. But this contradicts the definition of M' since $|M \oplus M'''| < |M \oplus M'|$.

From the preceding arguments, we deduce that either $M = M'$ or $M \oplus M'$ is a path of positive, even length that has u as an endpoint. Equivalently, $M \oplus M'$ is the edge set of an even-length path that has u as an endpoint. We claim that this path, with edges directed away from endpoint u, is a suitable choice for the directed path P claimed to exist in the statement of the lemma. Let P denote this directed path, and let $\mathcal{P}$ denote $\mathit{paths}(A, u, M)$. Below we argue that P belongs to $\mathcal{P}$.

Let G denote $\mathit{digraph}(A, u, M)$, and let u' denote the bidder at which path P terminates. (If P is of length zero, then $u' = u$.) Observe that the total weight of the edges on path P is equal to $w(M) - w(M')$, and hence is equal to $w(A) - w(A')$. Furthermore, P is a shortest path in G from u to u', because if there is a shorter path Q in G from u to u', then $\mathit{augment}(A, u, M, Q)$ is an MCM of A' that is heavier than M', a contradiction since M' is an MWMCM of A'.

Assume, for the sake of contradiction, that P does not belong to $\mathcal{P}$. Let Q be a directed path in $\mathcal{P}$, and let M'' denote $\mathit{augment}(A, u, M, Q)$. By the definition of $\mathcal{P}$, we deduce that $w(Q) \leq w(P)$. On the other hand, we claim that $w(P) \leq w(Q)$; if not, M'', which is an MCM of A', has a higher weight than M', a contradiction since M' is an MWMCM of A'. Hence $w(P) = w(Q)$ and thus $w(M') = w(M'')$. Let u'' denote the bidder at which path Q terminates. Since $w(P) = w(Q)$, Q belongs to $\mathcal{P}$, and P does not belong to $\mathcal{P}$, we deduce that u'' has a lower priority than u'. But then M'', which is an MWMCM of A' since $w(M') = w(M'')$, has a higher priority than M', a contradiction since M' is a greedy MWMCM of A'. Thus we conclude that P belongs to $\mathcal{P}$, and the definition of $\mathit{augment}(A, u, M, P)$ implies that $M' = \mathit{augment}(A, u, M, P)$. □

Lemma 36. Let $A = (U, V)$ and $A' = A + u$ be UAPs, let M be a greedy MWMCM of A, let P denote a path in $\mathit{paths}(A, u, M)$, and let M' denote $\mathit{augment}(A, u, M, P)$. Then M' is a greedy MWMCM of A'.

Proof. Let M^* be a greedy MWMCM of A'. By Lemma 35, there is a path Q in $paths(A, u, M)$ such that $M^* = augment(A, u, M, Q)$. Using the definition of $paths(A, u, M)$, we deduce that M' is an MCM of A' with $w(M') = w(M^*)$ and $priority(M') = priority(M^*)$. It follows that M' is a greedy MWMCM of A'. $\square$