On the Existence of Three-Dimensional Stable Matchings with Cyclic Preferences

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Abstract. We study the three-dimensional stable matching problem with cyclic preferences. This model involves three types of agents, with an equal number of agents of each type. The types form a cyclic order such that each agent has a complete preference list over the agents of the next type. We consider the open problem of the existence of three-dimensional matchings in which no triple of agents prefer each other to their partners. Such matchings are said to be weakly stable. We show that contrary to published conjectures, weakly stable three-dimensional matchings need not exist. Furthermore, we show that it is NP-complete to determine whether a weakly stable three-dimensional matching exists. We achieve this by reducing from the variant of the problem where preference lists are allowed to be incomplete. Our results can be generalized to the k-dimensional stable matching problem with cyclic preferences for $k \geq 3$.

Keywords: Stable matching · Three-dimensional matching · NP-completeness.

1 Introduction

The study of stable matchings was started by Gale and Shapley [9], who investigated a market with two types of agents. The two-dimensional stable matching problem involves an equal number of men and women, each of whom has a complete preference list over the agents of the opposite sex. The goal is to find a matching between the men and the women such that no man and woman prefer each other to their partners. Matchings satisfying this property are said to be stable. Gale and Shapley showed that a solution for the two-dimensional stable matching problem always exists and can be computed in polynomial time. Their result also applies to the variant where preference lists may be incomplete due to unacceptable partners, and the number of men may be different from the number of women.

The problem of generalizing stable matchings to markets with three types of agents was posed by Knuth [13]. In pursuit of an existence theorem and an elegant theory analogous to those of the Gale-Shapley model, the three-dimensional stable matching problem has been studied with respect to a number of preference structures. When each agent has preferences over pairs of agents from the other two types, stable matchings need not exist [1, 16]. Furthermore, it is NP-complete to determine whether a stable matching exists [16, 18], even if the preferences are consistent with product orders [11]. When two types of agents care primarily about each other and secondarily about the remaining type, a stable matching always exists and can be obtained by computing two-dimensional stable matchings using the Gale-Shapley algorithm in a hierarchical manner [5]. When the types form a cyclic order such that each type of agent cares primarily about the next type and secondarily about the other type, stable matchings need not exist [3].

A prominent problem mentioned in several of the aforementioned papers [3, 11, 16] is the three-dimensional stable matching problem for the case where the types form a cyclic order such that each type of agent cares only about the next type and not the other type. Following the terminology of the survey of Manlove [15], we call this the three-dimensional stable matching problem with cyclic preferences (3-DSM-CYC), and refer to the three types of agents as men, women, and dogs. A number of stability notions [11] can be considered in 3-DSM-CYC. In this paper, we focus on weak stability, which is the most permissive one and has received the most attention in the literature. It is known that determining whether a 3-DSM-CYC instance has a strongly stable matching is NP-complete [2]. For the variant where ties are allowed, determining the existence of a super-stable matching is also NP-complete [12]. However, it remained an open problem for weakly stable matchings in 3-DSM-CYC.

In 3-DSM-CYC, there are an equal number of men, women, and dogs. Each man has a complete preference list over the women, each woman has a complete preference list over the dogs, and each dog has a complete preference list over the men. A family is a triple consisting of a man, a woman, and a dog. A matching is a set of agent-disjoint families. A family is strongly blocking if every agent in the family prefers each other to their partners in the matching. A matching is weakly stable if it admits no strongly blocking family. This problem is related to applications such as kidney exchange [2] and three-sided network services [4].

The formulation of 3-DSM-CYC first appeared in the paper of Ng and Hirschberg [16], where it is attributed to Knuth. Using a greedy approach, Boros et al. [3] showed that every 3-DSM-CYC instance with at most 3 agents per type has a weakly stable matching. Their result also applies to the k-dimensional generalization of the problem, which we call k-DSM-CYC. For $k \geq 3$, they showed that every k-DSM-CYC instance with at most k agents per type has a weakly stable matching. Using a case analysis, Eriksson et al. [6] showed that every 3-DSM-CYC instance with at most 4 agents per type has a weakly stable matching, and they conjectured that every 3-DSM-CYC instance has a weakly stable matching. In fact, they posed the stronger conjecture that for a certain "strongest link" generalization of 3-DSM-CYC, every instance with at least two agents per type has at least two weakly stable matchings. Eriksson et al. also investigated and ruled out the use of certain arguments based on "effectivity functions" and "balanced games" for proving the 3-DSM-CYC conjecture. Using an efficient greedy procedure, Hofbauer [10] showed that for $k \geq 3$, every k-DSM-CYC instance with at most $k+1$ agents per type has a weakly stable matching. Using a satisfiability problem formulation and an extensive computer-assisted search, Pashkovich and Poirrier [17] showed that every 3-DSM-CYC instance with exactly 5 agents per type has at least two weakly stable matchings. Escamocher and O'Sullivan [7] showed that the number of weakly stable matchings is exponential in the size of the 3-DSM-CYC instance if agents of the same type are restricted to have the same preferences. They also conjectured that for unrestricted 3-DSM-CYC instances, there are exponentially many weakly stable matchings.

Hardness results are known for some related problems. For the variant of 3-DSM-CYC where preference lists are allowed to be incomplete, which we refer to as 3-DSMI-CYC, Biró and McDermid [2] showed that determining whether a weakly stable matching exists is NP-complete. Farczadi et al. [8] showed that determining whether a given perfect two-dimensional matching can be extended to a three-dimensional weakly stable matching in 3-DSM-CYC is also NP-complete. However, the existence of weakly stable matchings in 3-DSM-CYC remained unresolved. Manlove [15] described it as an "intriguing open problem", and Woeginger [19] classified it as "hard and outstanding".

Our Techniques and Contributions. In this paper, we show that there exists a 3-DSM-CYC instance that has no weakly stable matching. This disproves the conjectures of Eriksson et al. [6] and Escamocher and O'Sullivan [7]. Furthermore, we show that determining whether a 3-DSM-CYC instance has a weakly stable matching is NP-complete. We achieve this by reducing from the problem of determining whether a 3-DSMI-CYC instance has a weakly stable matching. Our results generalize to k-DSM-CYC for $k \geq 3$.

Our main technique involves converting each agent in 3-DSMI-CYC to a gadget consisting of one non-dummy agent and many dummy agents. The dummy agents in our gadget give rise to chains of admirers. (See Remark 2 in Section 4.3.) By applying the weak stability condition to the chains of admirers, we are able to obtain some control over the partner of the non-dummy agent.

Organization of This Paper. In Section 2, we present the formal definitions of k-DSM-CYC and k-DSMI-CYC. In Section 3, we show that the NP-completeness result of Biró and McDermid [2] can be extended to k-DSMI-CYC. In Section 4, we show that k-DSM-CYC is NP-complete by a reduction from k-DSMI-CYC. In Section 5, we conclude by mentioning some potential future work.

2 Preliminaries

In this paper, we use $\langle z \in Z \mid \mathcal{P}(z) \rangle$ to denote the list of all tuples $z \in Z$ satisfying predicate $\mathcal{P}(z)$, where the tuples are sorted in increasing lexicographical order. Given two lists Y and Z, we denote their concatenation as $Y \cdot Z$. For any $k \geq 1$, we use $\oplus_k$ to denote addition modulo k.

2.1 The Models

Let $k \geq 2$. The k-dimensional stable matching problem with incomplete lists and cyclic preferences (k-DSMI-CYC) involves a finite set $A = I \times \{0, \ldots, k-1\}$ of agents, where each agent $\alpha = (i, t) \in A$ is associated with an identifier i and a type t. (When $k = 3$, we can think of the sets $I \times \{0\}$, $I \times \{1\}$, and $I \times \{2\}$ as the sets of men, women, and dogs, respectively.) Each agent $\alpha = (i, t) \in A$ has a strict preference list P_α over a subset of agents of type $t' = t \oplus_k 1$. In other words, every agent in $I \times \{t \oplus_k 1\}$ appears in P_α at most once, and every element in P_α belongs to $I \times \{t \oplus_k 1\}$. For every $\alpha, \alpha', \alpha'' \in A$, we say that α prefers α' to α'' if α' appears in P_α and either agent α'' appears in P_α after α' or agent α'' does not appear in P_α. We denote this k-DSMI-CYC instance as $X = (A, \{P_\alpha\}_{\alpha \in A})$.

Given a k-DSMI-CYC instance $X = (A, \{P_\alpha\}_{\alpha \in A})$, a *family* is a tuple

$$(\alpha_0, \ldots, \alpha_{k-1}) \in A^k$$

such that $\alpha_t \in I \times \{t\}$ and $\alpha_{t \oplus_k 1}$ appears in P_{α_t} for every $t \in \{0, \ldots, k-1\}$. A *matching* μ is a set of agent-disjoint families. In other words, for every $t, t' \in \{0, \ldots, k-1\}$ and $(\alpha_0, \ldots, \alpha_{k-1}), (\alpha'_0, \ldots, \alpha'_{k-1}) \in \mu$, if $\alpha_t = \alpha'_t$, then $\alpha_{t'} = \alpha'_{t'}$. Given a matching μ and an agent $\alpha \in A$, if $\alpha = \alpha_t$ for some $(\alpha_0, \ldots, \alpha_{k-1}) \in \mu$ and $t \in \{0, \ldots, k-1\}$, we say that α is matched to $\alpha_{t \oplus_k 1}$, and we write $\mu(\alpha) = \alpha_{t \oplus_k 1}$. Otherwise, we say that α is unmatched, and we write $\mu(\alpha) = \alpha$.

Given a matching μ, we say that a family $(\alpha_0, \ldots, \alpha_{k-1})$ is *strongly blocking* if α_t prefers $\alpha_{t \oplus_k 1}$ to $\mu(\alpha_t)$ for every $t \in \{0, \ldots, k-1\}$. A matching μ is *weakly stable* if it does not admit any strongly blocking family.

The k-dimensional stable matching problem with cyclic preferences (k-DSM-CYC) is defined as the special case of k-DSMI-CYC in which every agent in $I \times \{t \oplus_k 1\}$ appears exactly once in P_α for every agent $\alpha = (i, t) \in A$.

Notice that when incomplete lists are allowed, the case of an unequal number of agents of each type can be handled within our k-DSMI-CYC model by padding with dummy agents whose preference lists are empty. Hence, the results of Biró and McDermid [2] apply to our 3-DSMI-CYC model. When preference lists are complete, we follow the literature and focus on the case where each type has an equal number of agents. Our result shows that even when restricted to the case of an equal number of agents of each type, a given k-DSM-CYC instance need not admit a weakly stable matching, and determining the existence of a weakly stable matching is NP-complete.

2.2 Polynomial-Time Verification

Given a matching μ of a k-DSMI-CYC instance with n agents per type, it is straightforward to determine whether μ is weakly stable in $O(n^k)$ time by checking that none of the $O(n^k)$ families is strongly blocking. The following theorem shows that when k is large, there is a more efficient method to determine whether a given matching is weakly stable. A proof is provided in [14].

Theorem 1. *There exists a* poly(n, k)*-time algorithm to determine whether a given matching* μ *is weakly stable for a* k-DSMI-CYC *instance, where* n *is the number of agents per type.*

3 NP-Completeness of k-DSMI-CYC

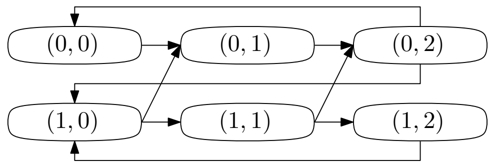
In this section, we show that for every $k \geq 3$, it is NP-complete to determine whether a given k-DSMI-CYC instance has a weakly stable matching. We achieve this by reducing from the problem of determining whether a 3-DSMI-CYC instance has a weakly stable matching.

3.1 The Reduction

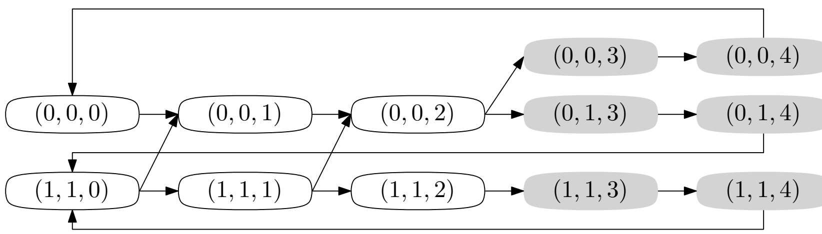
Let $k \geq 4$. Consider an input 3-DSMI-CYC instance $X = (A, \{P_\alpha\}_{\alpha \in A})$ where $A = I \times \{0, 1, 2\}$. Our reduction constructs a k-DSMI-CYC instance $\hat{X} = (\hat{A}, \{\hat{P}_{\hat{\alpha}}\}_{\hat{\alpha} \in \hat{A}})$ as follows.

- Let $\hat{I} = I \times I$ and $\hat{A} = I \times I \times \{0, \ldots, k-1\}$. For every agent $(i, t) \in A$, we call $\hat{\alpha} = (i, i, t) \in \hat{A}$ the non-dummy agent corresponding to (i, t). We call the agents
$$\{(i, j, t) \in \hat{A} \mid t \notin \{0, 1, 2\} \text{ or } i \neq j\}$$
dummy agents.
- For every agent $\hat{\alpha} = (i, j, t) \in \hat{A}$, we construct the preference list $\hat{P}_{\hat{\alpha}}$ as follows.
  - If $0 \leq t \leq 1$ and $i = j$, we list in $\hat{P}_{\hat{\alpha}}$ the agents
  $$\{(i', j', t') \in I \times I \times \{t+1\} \mid i' = j' \text{ and } (i', t') \text{ is in } P_{(i,t)}\}$$
  in the order in which the corresponding agent (i', t') appears in $P_{(i,t)}$.
  - If $t = 2$ and $i = j$, we list in $\hat{P}_{\hat{\alpha}}$ the agents
  $$\{(i', j', t') \in I \times I \times \{3\} \mid i' = i \text{ and } (j', 0) \text{ is in } P_{(i,2)}\}$$
  in the order in which the corresponding agent $(j', 0)$ appears in $P_{(i,2)}$.
  - If $0 \leq t \leq 2$ and $i \neq j$, we define $\hat{P}_{\hat{\alpha}}$ as the empty list.
  - If $3 \leq t \leq k-2$ and $(j, 0)$ is in $P_{(i,2)}$, we define $\hat{P}_{\hat{\alpha}}$ as $\langle (i, j, t+1) \rangle$.
  - If $t = k-1$ and $(j, 0)$ is in $P_{(i,2)}$, we define $\hat{P}_{\hat{\alpha}}$ as $\langle (j, j, 0) \rangle$.
  - If $3 \leq t \leq k-1$ and $(j, 0)$ is not in $P_{(i,2)}$, we define $\hat{P}_{\hat{\alpha}}$ as the empty list.

Figure 1 shows an example of the reduction when $k = 5$ and $I = \{0, 1\}$.



(a) The input 3-DSMI-CYC instance.



(b) The output 5-DSMI-CYC instance. Agents with empty preference lists are omitted.

Fig. 1. Example of a reduction from 3-DSMI-CYC to 5-DSMI-CYC. An arrow indicates that the target agent appears in the preference list of the source agent.

3.2 Correctness of the Reduction

Proofs of the three claims stated below are provided in [14]. We emphasize that the important special case of Theorem 2 where $k = 3$ is due to Biró and McDermid [2, Lemma 1].

Lemma 1. *Let $k \geq 4$. Consider the reduction given in Section 3.1. The output k-*DSMI-CYC *instance $\hat{X}$ has a weakly stable matching if and only if the input 3-*DSMI-CYC *instance X has a weakly stable matching.*

Theorem 2. *Let $k \geq 3$. Then there exists a k-*DSMI-CYC *instance that has no weakly stable matching.*

Theorem 3. *Let $k \geq 3$. Then it is NP-complete to determine whether a k-*DSMI-CYC *instance has a weakly stable matching.*

4 NP-Completeness of k-DSM-CYC

In this section, we show that for every $k \geq 3$, it is NP-complete to determine whether a k-DSM-CYC instance has a weakly stable matching. We achieve this by reducing from the problem of determining whether a k-DSMI-CYC instance has a weakly stable matching. Since the dimensions of both the input instance and the output instance of the reduction are equal to k, throughout this section, we write $\oplus$ instead of $\oplus_k$ for better readability.

4.1 The Reduction

Let $k \geq 3$. Consider an input k-DSMI-CYC instance $X = (A, \{P_\alpha\}_{\alpha \in A})$ where $A = I \times \{0, \ldots k-1\}$. We may assume that $I = \{0, \ldots, |I|-1\}$, so agents in A can be compared lexicographically. Our reduction constructs a k-DSM-CYC instance $\hat{X} = (\hat{A}, \{\hat{P}_{\hat{\alpha}}\}_{\hat{\alpha} \in \hat{A}})$ as follows.

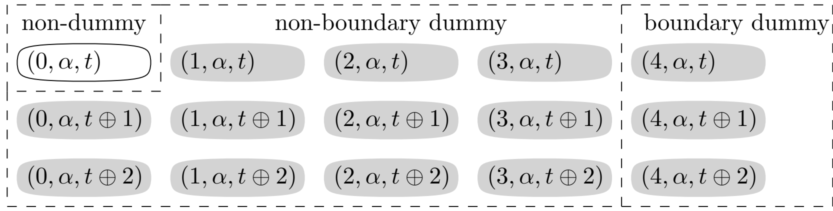
- Let $J = \{0, \ldots, (k-1)^2\}$. Let $\hat{I} = J \times A$ and $\hat{A} = J \times A \times \{0, \ldots, k-1\}$. For every agent $\alpha \in A$, we call $J \times \{\alpha\} \times \{0, \ldots, k-1\}$ the gadget corresponding to α.
- For every agent $\hat{\alpha} = (j, \alpha, t) \in \hat{A}$ such that $j = 0$ and $\alpha \in I \times \{t\}$, we call $\hat{\alpha}$ the non-dummy agent corresponding to α. Let $\hat{P}'_\alpha$ be the list obtained by replacing every α' in P_α by $(0, \alpha', t \oplus 1)$. We define the preference list $\hat{P}_{\hat{\alpha}}$ as $\hat{P}'_\alpha \cdot \langle (j', \alpha', t') \in J \times A \times \{t \oplus 1\} \mid \alpha' = \alpha \rangle$ followed by the remaining agents in $J \times A \times \{t \oplus 1\}$ in an arbitrary order.
- For every agent $\hat{\alpha} = (j, \alpha, t) \in \hat{A}$ such that $j = (k-1)^2$, we call $\hat{\alpha}$ a boundary dummy agent, and we define the preference list $\hat{P}_{\hat{\alpha}}$ as
$$\langle (j', \alpha', t') \in J \times A \times \{t \oplus 1\} \mid \alpha' = \alpha \text{ and } j' < (k-1)^2 \rangle \cdot \langle (j', \alpha', t') \in J \times A \times \{t \oplus 1\} \mid j' = (k-1)^2 \rangle$$
followed by the remaining agents in $J \times A \times \{t \oplus 1\}$ in an arbitrary order.
- For every agent $\hat{\alpha} = (j, \alpha, t) \in \hat{A}$ such that $(j, \alpha, t) \notin \{0\} \times (I \times \{t\}) \times \{t\}$ and $j < (k-1)^2$, we call $\hat{\alpha}$ a non-boundary dummy agent, and we define the preference list $\hat{P}_{\hat{\alpha}}$ as $\langle (j', \alpha', t') \in J \times A \times \{t \oplus 1\} \mid \alpha' = \alpha \rangle$ followed by the remaining agents in $J \times A \times \{t \oplus 1\}$ in an arbitrary order.

As shown in Figure 2(a), the gadget corresponding to $\alpha \in I \times \{t\}$ can be visualized as a grid of agents with k rows and $(k-1)^2 + 1$ columns. The non-boundary dummy agents in the same row have essentially the same preferences, which begin with the agents in the next row from left to right. The preferences of the boundary dummy agents are similar to those of the non-boundary dummy agents, except that they incorporate the other boundary dummy agents in a special manner. Meanwhile, the preferences of the non-dummy agent $(0, \alpha, t)$ reflect the preferences of agent α by starting with $\hat{P}'_\alpha$.

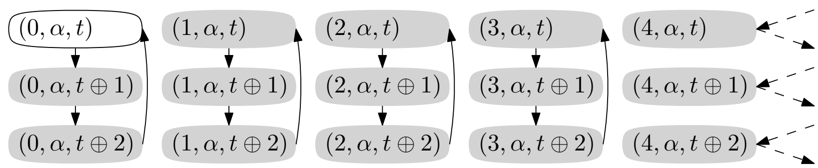
Remark 1. The reason our gadget has $(k-1)^2 + 1$ columns will become clearer when we present Lemmas 4 and 5 below. At a high level, Lemma 4 is invoked $k-1$ times within the proof of Lemma 5, and each such invocation leads to an increase in the number of columns of $k-1$.

4.2 Correctness of the Reduction

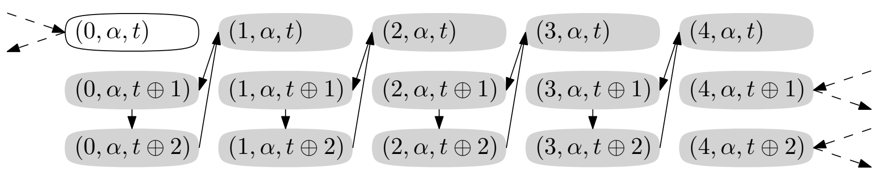
Lemmas 2 and 3 below show that the reduction in Section 4.1 is a correct reduction from k-DSMI-CYC to k-DSM-CYC. The associated proofs are presented in Sections 4.4 and 4.5.



(a) The structure of the gadget.



(b) The matching $\hat{\mu}$ induced by μ when α is unmatched in μ.



(c) The matching $\hat{\mu}$ induced by μ when α is matched in μ.

Fig. 2. Example of a gadget corresponding to $\alpha \in I \times \{t\}$ when $k = 3$. An arrow indicates that the source agent is matched to the target agent.

Lemma 2. *Let $k \geq 3$. Consider the reduction given in Section 4.1. If the input k-*DSMI-CYC *instance X has no weakly stable matching, then the output k-*DSM-CYC *instance $\hat{X}$ has no weakly stable matching.*

Lemma 3. *Let $k \geq 3$. Consider the reduction in Section 4.1. If the input k-*DSMI-CYC *instance X has a weakly stable matching, then the output k-*DSM-CYC *instance $\hat{X}$ has a weakly stable matching.*

Proofs of the next two theorems are provided in [14].

Theorem 4. *Let $k \geq 3$. Then there exists a k-*DSM-CYC *instance that has no weakly stable matching.*

Theorem 5. *Let $k \geq 3$. Then it is NP-complete to determine whether a k-*DSM-CYC *instance has a weakly stable matching.*

4.3 Properties of the Gadget

In this subsection, we study the properties of the gadget in the scenario that the non-dummy agent is not matched to a non-dummy agent corresponding to an acceptable partner. In Lemma 4, we show that in this scenario, many agents in the gadget are matched to agents in the same gadget. In Lemma 5, we apply Lemma 4 inductively to show that in the same scenario, every agent in the same family as the non-dummy agent belongs to the same gadget.

Remark 2. In the proof of Lemma 4 below, we can think of $\hat{\alpha}_0, \ldots, \hat{\alpha}_{k-1}$ as a chain of admirers in the gadget corresponding to α, where $\hat{\alpha}_s$ prefers $\hat{\alpha}_{s+1}$ to $\hat{\mu}(\hat{\alpha}_s)$. By applying the weak stability condition to this chain of admirers, we show that $\hat{\alpha}_{k-1}$ is matched to a partner no worse than $\hat{\alpha}_0$.

Lemma 4. *Let $\hat{\mu}$ be a weakly stable matching in $\hat{X}$. Let $t^* \in \{0, \ldots, k-1\}$ and $\alpha^* \in I \times \{t^*\}$ such that $\hat{\mu}(0, \alpha^*, t^*)$ is not in $\hat{P}'_{\alpha^*}$. Let $t \in \{0, \ldots, k-1\}$ and $j \in J$ such that $j \leq (k-1) \cdot (k-2)$. Then $\hat{\mu}(j, \alpha^*, t) \in \{0, \ldots, j+k-1\} \times \{\alpha^*\} \times \{t \oplus 1\}$.*

Proof. Let $\hat{A}_s = \langle (j', \alpha', t') \in J \times \{\alpha^*\} \times \{t \oplus s \oplus 1\} \mid j' \leq j + k - s - 1 \rangle$ for every $s \in \{0, \ldots, k-1\}$. For the sake of contradiction, suppose $\hat{\mu}(j, \alpha^*, t)$ is not in $\hat{A}_0$.

For every $s \in \{0, \ldots, k-2\}$, since the length of $\hat{A}_s$ is greater than the length of $\hat{A}_{s+1}$, there exists $\hat{\alpha}_s$ in $\hat{A}_s$ such that $\hat{\mu}(\hat{\alpha}_s)$ is not in $\hat{A}_{s+1}$. Let $\hat{\alpha}_{k-1} = (j, \alpha^*, t)$. Then $\hat{\alpha}_{k-1}$ is in $\hat{A}_{k-1}$ and $\hat{\mu}(\hat{\alpha}_{k-1})$ is not in $\hat{A}_0$. Since $\hat{\mu}$ is a weakly stable matching of $\hat{X}$, the family $(\hat{\alpha}_{k-t-1}, \ldots, \hat{\alpha}_{(k-t-1) \oplus (k-1)})$ is not strongly blocking. So there exists $s^* \in \{0, \ldots, k-1\}$ such that $\hat{\alpha}_{s^*}$ does not prefer $\hat{\alpha}_{s^* \oplus 1}$ to $\mu(\hat{\alpha}_{s^*})$. Since $\hat{\alpha}_{s^*}$ is in $\hat{A}_{s^*}$, there exists $j^* \leq j + k - s^* - 1$ such that $\hat{\alpha}_{s^*} = (j^*, \alpha^*, t \oplus s^* \oplus 1)$. We consider two cases.

Case 1: $j^* = 0$ and $t \oplus s^* \oplus 1 = t^*$. Then $\hat{\alpha}_{s^*} = (0, \alpha^*, t^*)$ is a non-dummy agent and $\hat{P}'_{\alpha^*} \cdot \hat{A}_{s^* \oplus 1}$ is a prefix of the preference list $\hat{P}_{\hat{\alpha}_{s^*}}$. Since $\mu(\hat{\alpha}_{s^*})$ is not in $\hat{P}'_{\alpha^*} \cdot \hat{A}_{s^* \oplus 1}$ and $\hat{\alpha}_{s^* \oplus 1}$ is in $\hat{A}_{s^* \oplus 1}$, agent $\hat{\alpha}_{s^*}$ prefers $\hat{\alpha}_{s^* \oplus 1}$ to $\mu(\hat{\alpha}_{s^*})$, a contradiction.

Case 2: $j^* \neq 0$ or $t \oplus s^* \oplus 1 \neq t^*$. We consider two subcases.

Case 2.1: $j^* = (k-1)^2$. Since $(k-1)^2 = j^* \leq j + k - s^* - 1 \leq (k-1)^2 - s^*$, we have $s^* = 0$. Hence $\hat{\alpha}_0 = ((k-1)^2, \alpha^*, t \oplus 1)$ is a boundary dummy agent and $\hat{A}_1$ is a prefix of the preference list $\hat{P}_{\hat{\alpha}_0}$. Since $\mu(\hat{\alpha}_0)$ is not in $\hat{A}_1$ and $\hat{\alpha}_1$ is in $\hat{A}_1$, agent $\hat{\alpha}_0$ prefers $\hat{\alpha}_1$ to $\mu(\hat{\alpha}_0)$, a contradiction.

Case 2.2: $j^* < (k-1)^2$. Then $\hat{\alpha}_{s^*}$ is a non-boundary dummy agent and $\hat{A}_{s^* \oplus 1}$ is a prefix of the preference list $\hat{P}_{\hat{\alpha}_{s^*}}$. Since $\mu(\hat{\alpha}_{s^*})$ is not in $\hat{A}_{s^* \oplus 1}$ and $\hat{\alpha}_{s^* \oplus 1}$ is in $\hat{A}_{s^* \oplus 1}$, agent $\hat{\alpha}_{s^*}$ prefers $\hat{\alpha}_{s^* \oplus 1}$ to $\mu(\hat{\alpha}_{s^*})$, a contradiction. □

Lemma 5. *Let $\hat{\mu}$ be a weakly stable matching in $\hat{X}$. Let $j_0, \ldots, j_{k-1} \in J$ and $\alpha_0, \ldots, \alpha_{k-1} \in A$ such that $((j_0, \alpha_0, 0), \ldots, (j_{k-1}, \alpha_{k-1}, k-1)) \in \hat{\mu}$. Let $t^* \in \{0, \ldots, k-1\}$ such that $j_{t^*} = 0$ and $\alpha_{t^*} \in I \times \{t^*\}$. Suppose that $(j_{t^* \oplus 1}, \alpha_{t^* \oplus 1}, t^* \oplus 1)$ is not in $\hat{P}'_{\alpha_{t^*}}$. Then, for every $s \in \{0, \ldots, k-1\}$, we have $\alpha_{t^* \oplus s} = \alpha_{t^*}$ and $j_{t^* \oplus s} \leq (k-1) \cdot s$.*

Proof. We prove the claim by induction on s. When $s = 0$, we have $\alpha_{t^* \oplus s} = \alpha_{t^* \oplus 0} = \alpha_{t^*}$ and $j_{t^* \oplus s} = j_{t^*} = 0 \leq (k-1) \cdot s$.

Suppose $\alpha_{t^* \oplus (s-1)} = \alpha_{t^*}$ and $j_{t^* \oplus (s-1)} \leq (k-1) \cdot (s-1)$, where $s \in \{1, \ldots, k-1\}$. Since $(j_{t^* \oplus 1}, \alpha_{t^* \oplus 1}, t^* \oplus 1)$ is not in $\hat{P}'_{\alpha_{t^*}}$, agent $\hat{\mu}(0, \alpha_{t^*}, t^*)$ is not in $\hat{P}'_{\alpha_{t^*}}$. Let $t = t^* \oplus (s-1)$. Then $\alpha_t = \alpha_{t^* \oplus (s-1)} = \alpha_{t^*}$ and $j_t = j_{t^* \oplus (s-1)} \leq (k-1) \cdot (s-1) \leq (k-1) \cdot (k-2)$. So Lemma 4 implies that $\hat{\mu}(j_t, \alpha_{t^*}, t) \in \{0, \ldots, j_t + k - 1\} \times \{\alpha_{t^*}\} \times \{t \oplus 1\}$. Hence $j_{t \oplus 1} \leq j_t + k - 1$ and $\alpha_{t \oplus 1} = \alpha_{t^*}$, since $\hat{\mu}(j_t, \alpha_{t^*}, t) = \hat{\mu}(j_t, \alpha_t, t) = (j_{t \oplus 1}, \alpha_{t \oplus 1}, t \oplus 1)$. Thus $\alpha_{t^* \oplus s} = \alpha_{t \oplus 1} = \alpha_{t^*}$ and $j_{t^* \oplus s} = j_{t \oplus 1} \leq j_t + k - 1 = j_{t^* \oplus (s-1)} + k - 1 \leq (k-1) \cdot (s-1) + k - 1 = (k-1) \cdot s$. □

4.4 Proof of Lemma 2

The goal of this subsection is to prove Lemma 2. It suffices to show that every weakly stable matching $\hat{\mu}$ in $\hat{X}$ induces a weakly stable matching μ in X.

Recall that each agent in A has a corresponding non-dummy agent in $\hat{A}$, and that a family in X is a tuple of k agents in A such that each agent appears in the preference list of another. Hence we include in μ a family of agents in X whenever the corresponding family of non-dummy agents are matched in $\hat{\mu}$. More formally, we define the matching μ in X induced by $\hat{\mu}$ in $\hat{X}$ as the set of families $(\alpha_0, \ldots, \alpha_{k-1})$ in X satisfying $((0, \alpha_0, 0), \ldots, (0, \alpha_{k-1}, k-1)) \in \hat{\mu}$. Notice that every μ induced by a matching $\hat{\mu}$ in $\hat{X}$ is a valid matching in X since agent-disjoint families in $\hat{X}$ induce agent-disjoint families in X.

Lemma 6 below shows that if $\hat{\mu}$ is weakly stable and matches a non-dummy agent to a non-dummy agent corresponding to an acceptable partner, then μ matches the corresponding agents. Our proof relies on Lemma 5 and the weak stability of $\hat{\mu}$. Notice that if $\hat{\mu}$ is not weakly stable, it may be the case that $\hat{\mu}$ matches a family consisting of $k-1$ non-dummy agents and one dummy agent. In such a case, the corresponding $k-1$ agents are unmatched in the induced matching μ.

Lemma 6. *Let μ be the matching in X induced by a weakly stable matching $\hat{\mu}$ in $\hat{X}$. Let $t \in \{0, \ldots, k-1\}$ and $\alpha \in I \times \{t\}$ such that $\hat{\mu}(0, \alpha, t)$ is in $\hat{P}'_\alpha$. Then $\hat{\mu}(0, \alpha, t) = (0, \mu(\alpha), t \oplus 1)$.*

Proof. For the sake of contradiction, suppose $\hat{\mu}(0, \alpha, t) \neq (0, \mu(\alpha), t \oplus 1)$. Since $\hat{\mu}(0, \alpha, t)$ is in $\hat{P}'_\alpha$, we have $((j_0, \alpha_0, 0), \ldots, (j_{k-1}, \alpha_{k-1}, k-1)) \in \hat{\mu}$ for some $j_0, \ldots, j_{k-1} \in J$ and $\alpha_0, \ldots, \alpha_{k-1} \in A$ such that $(j_t, \alpha_t, t) = (0, \alpha, t)$ and $(j_{t \oplus 1}, \alpha_{t \oplus 1}, t \oplus 1)$ is in $\hat{P}'_\alpha$. Let

$$T = \{t' \in \{0, \ldots, k-1\} \mid \alpha_{t'} \in I \times \{t'\} \text{ and } (j_{t' \oplus 1}, \alpha_{t' \oplus 1}, t' \oplus 1) \text{ is in } \hat{P}'_{\alpha_{t'}}\}.$$

Then $t \in T$. We consider two cases.

Case 1: $T = \{0, \ldots, k-1\}$. Then for every $t' \in T = \{0, \ldots, k-1\}$, we have $\alpha_{t'} \in I \times \{t'\}$ and $(j_{t' \oplus 1}, \alpha_{t' \oplus 1}, t' \oplus 1)$ is in $\hat{P}'_{\alpha_{t'}}$. So $j_{t' \oplus 1} = 0$ and $\alpha_{t' \oplus 1}$ is in $P_{\alpha_{t'}}$ for every $t' \in \{0, \ldots, k-1\}$. Hence $(\alpha_0, \ldots, \alpha_{k-1})$ is a valid family in X. Since μ is induced by $\hat{\mu}$ and $((0, \alpha_0, 0), \ldots, (0, \alpha_{k-1}, k-1)) \in \hat{\mu}$, we have $(\alpha_0, \ldots, \alpha_{k-1}) \in \mu$. Thus $\mu(\alpha) = \mu(\alpha_t) = \alpha_{t\oplus 1}$, which contradicts $(0, \mu(\alpha), t \oplus 1) \neq \hat{\mu}(0, \alpha, t) = (0, \alpha_{t\oplus 1}, t \oplus 1)$.

Case 2: $T \neq \{0, \ldots, k-1\}$. Then there exists a smallest $s^* \in \{1, \ldots, k-1\}$ such that $t \oplus s^* \notin T$. Then $t \oplus (s^* - 1) \in T$. Let $t^* = t \oplus s^*$. Since $t^* \oplus (-1) = t \oplus (s^* - 1) \in T$, we have $\alpha_{t^* \oplus (-1)} \in I \times \{t^* \oplus (-1)\}$ and $(j_{t^*}, \alpha_{t^*}, t^*)$ is in $\hat{P}'_{\alpha_{t^* \oplus (-1)}}$. So $j_{t^*} = 0$ and α_{t^*} is in $P_{\alpha_{t^* \oplus (-1)}}$. Hence $\alpha_{t^*} \in I \times \{t^*\}$. Since $\alpha_{t^*} \in I \times \{t^*\}$ and $t^* = t \oplus s^* \notin T$, agent $(j_{t^* \oplus 1}, \alpha_{t^* \oplus 1}, t^* \oplus 1)$ is not in $\hat{P}'_{\alpha_{t^*}}$. So Lemma 5 implies $\alpha_{t^* \oplus (k-1)} = \alpha_{t^*}$. Hence $\alpha_{t^* \oplus (-1)} = \alpha_{t^* \oplus (k-1)} = \alpha_{t^*} \in I \times \{t^*\}$, which contradicts $\alpha_{t^* \oplus (-1)} \in I \times \{t^* \oplus (-1)\}$. □

Proof of Lemma 2. For the sake of contradiction, suppose X has no weakly stable matching and $\hat{X}$ has a weakly stable matching $\hat{\mu}$. Let μ be the matching in X induced by $\hat{\mu}$.

Since μ is not a weakly stable matching of X, there exists a strongly blocking family $(\alpha_0, \ldots, \alpha_{k-1})$. Since $\hat{\mu}$ is a weakly stable matching of $\hat{X}$, the family

$$((0, \alpha_0, 0), \ldots, (0, \alpha_{k-1}, k-1))$$

is not strongly blocking. So there exists $t \in \{0, \ldots, k-1\}$ such that $(0, \alpha_t, t)$ does not prefer $(0, \alpha_{t\oplus 1}, t \oplus 1)$ to $\hat{\mu}(0, \alpha_t, t)$. Since $(\alpha_0, \ldots, \alpha_{k-1})$ is a family in X, agent $\alpha_{t\oplus 1}$ is in P_{α_t}. So $(0, \alpha_{t\oplus 1}, t \oplus 1)$ is in $\hat{P}'_{\alpha_t}$. Hence $\hat{\mu}(0, \alpha_t, t)$ appears in $\hat{P}'_{\alpha_t}$ no later than $(0, \alpha_{t\oplus 1}, t \oplus 1)$, since $\hat{P}'_{\alpha_t}$ is a prefix of the preference list $\hat{P}_{(0, \alpha_t, t)}$.

Since $\hat{\mu}(0, \alpha_t, t)$ is in $\hat{P}'_{\alpha_t}$, Lemma 6 implies $\hat{\mu}(0, \alpha_t, t) = (0, \mu(\alpha_t), t \oplus 1)$. Since $(0, \mu(\alpha_t), t \oplus 1)$ appears in $\hat{P}'_{\alpha_t}$ no later than $(0, \alpha_{t\oplus 1}, t \oplus 1)$, agent $\mu(\alpha_t)$ appears in P_{α_t} no later than $\alpha_{t\oplus 1}$. Hence α_t does not prefer $\alpha_{t\oplus 1}$ to $\mu(\alpha_t)$. So $(\alpha_0, \ldots, \alpha_{k-1})$ is not a strongly blocking family of μ, a contradiction. □

4.5 Proof of Lemma 3

The goal of this subsection is to prove Lemma 3. It suffices to show that every weakly stable matching μ in X induces a weakly stable matching $\hat{\mu}$ in $\hat{X}$. We construct the matching $\hat{\mu}$ induced by μ as follows.

- For every $(\alpha_0, \ldots, \alpha_{k-1}) \in \mu$, we include in $\hat{\mu}$ the family
$$((0, \alpha_0, 0), \ldots, (0, \alpha_{k-1}, k-1)).$$
- For every agent $\alpha \in A$ and $j \in J$ such that $j < (k-1)^2$, we include in $\hat{\mu}$ the family $((j + \delta_0(\alpha), \alpha, 0), \ldots, (j + \delta_{k-1}(\alpha), \alpha, k-1))$, where
$$\delta_t(\alpha) = \begin{cases} 1 & \text{if } \mu(\alpha) \neq \alpha \text{ and } \alpha \in I \times \{t\} \\ 0 & \text{otherwise} \end{cases}$$
- For every $t \in \{0, \ldots, k-1\}$, let R_t be the list
$$\langle (j', \alpha', t') \in \{(k-1)^2\} \times A \times \{t\} \mid \delta_{t'}(\alpha') = 0 \rangle.$$
We include in $\hat{\mu}$ the family $(R_0[s], \ldots, R_{k-1}[s])$ for every $0 \leq s < |A| - |\mu|$, where $R_t[s]$ denotes the $(s+1)$th element of R_t.

Figures 2(b) and 2(c) show the gadget under the matching $\hat{\mu}$.

It is straightforward to check that the families in $\hat{\mu}$ induced by a matching μ are agent-disjoint. Hence $\hat{\mu}$ is a valid matching in $\hat{X}$.

Lemma 7. *Let $\hat{\mu}$ be the matching in $\hat{X}$ induced by a matching μ in X. Let $t \in \{0, \ldots, k-1\}$ and $\alpha \in A$ such that $\alpha \in I \times \{t\}$. Let $j' \in J$ and $\alpha' \in A$ such that non-dummy agent $(0, \alpha, t)$ prefers $(j', \alpha', t \oplus 1)$ to $\hat{\mu}(0, \alpha, t)$. Then $(j', \alpha', t \oplus 1)$ is in $\hat{P}'_\alpha$ and α prefers α' to $\mu(\alpha)$.*

Proof. Notice that $\hat{P}'_\alpha \cdot \langle (0, \alpha, t \oplus 1) \rangle$ is a prefix of the preference list $\hat{P}_{(0,\alpha,t)}$ of non-dummy agent $(0, \alpha, t)$. We consider two cases.

Case 1: $\mu(\alpha) \neq \alpha$. Then $\hat{\mu}(0, \alpha, t) = (0, \mu(\alpha), t \oplus 1)$. Since $(0, \alpha, t)$ prefers $(j', \alpha', t \oplus 1)$ to $(0, \mu(\alpha), t \oplus 1)$, agent $(j', \alpha', t \oplus 1)$ appears in $\hat{P}'_\alpha$ before $(0, \mu(\alpha), t \oplus 1)$. Hence α prefers α' to $\mu(\alpha)$.

Case 2: $\mu(\alpha) = \alpha$. Then $\hat{\mu}(0, \alpha, t) = (0, \alpha, t \oplus 1)$. Since $(0, \alpha, t)$ prefers $(j', \alpha', t \oplus 1)$ to $(0, \alpha, t \oplus 1)$, agent $(j', \alpha', t \oplus 1)$ is in $\hat{P}'_\alpha$. Then α' is in P_α, and hence α prefers α' to $\mu(\alpha)$. □

Lemma 8. *Let $\hat{\mu}$ be the matching in $\hat{X}$ induced by a weakly stable matching μ in X. Let $j_0, \ldots, j_{k-1} \in J$ and $\alpha_0, \ldots, \alpha_{k-1} \in A$ such that*

$$((j_0, \alpha_0, 0), \ldots, (j_{k-1}, \alpha_{k-1}, k-1))$$

is a strongly blocking family of $\hat{\mu}$. Then $j_t - \delta_t(\alpha_t) \geq (k-1)^2$ for every $t \in \{0, \ldots, k-1\}$.

Proof. Let $t^* \in \{0, \ldots, k-1\}$ such that

$$j_{t^*} - \delta_{t^*}(\alpha_{t^*}) = \min_{t \in \{0, \ldots, k-1\}} (j_t - \delta_t(\alpha_t)).$$

For the sake of contradiction, suppose $j_{t^*} - \delta_{t^*}(\alpha_{t^*}) < (k-1)^2$. We consider two cases.

Case 1: $j_{t^*} = 0$ and $\alpha_{t^*} \in I \times \{t^*\}$. Let $T = \{t \mid j_t = 0 \text{ and } \alpha_t \in I \times \{t\}\}$. Then $t^* \in T$. We consider two subcases.

Case 1.1: $T = \{0, \ldots, k-1\}$. Then for every $t \in \{0, \ldots, k-1\} = T$, since $(0, \alpha_t, t)$ prefers $(0, \alpha_{t \oplus 1}, t \oplus 1)$ to $\hat{\mu}(0, \alpha_t, t)$, Lemma 7 implies that α_t prefers $\alpha_{t \oplus 1}$ to $\mu(\alpha_t)$. Hence $(\alpha_0, \ldots, \alpha_{k-1})$ is a strongly blocking family of μ, which contradicts the stability of μ.

Case 1.2: $\{t^*\} \subseteq T \subsetneq \{0, \ldots, k-1\}$. Then there exists s^* such that $s^* \in T$ and $s^* \oplus 1 \notin T$. Since $s^* \in T$, we have $j_{s^*} = 0$ and $\alpha_{s^*} \in I \times \{s^*\}$. Since $(0, \alpha_{s^*}, s^*)$ prefers $(j_{s^* \oplus 1}, \alpha_{s^* \oplus 1}, s^* \oplus 1)$ to $\hat{\mu}(0, \alpha_{s^*}, s^*)$, Lemma 7 implies that $(j_{s^* \oplus 1}, \alpha_{s^* \oplus 1}, s^* \oplus 1)$ is in $\hat{P}'_{\alpha_{s^*}}$. Hence $j_{s^* \oplus 1} = 0$ and $\alpha_{s^* \oplus 1} \in I \times \{s^* \oplus 1\}$, which contradicts $s^* \oplus 1 \notin T$.

Case 2: Either $j_{t^*} \neq 0$ or $\alpha_{t^*} \notin I \times \{t^*\}$. Thus $(j_{t^*}, \alpha_{t^*}, t^*)$ is a dummy agent. We consider two subcases.

Case 2.1: $j_{t^*} < (k-1)^2$. Since

$$\hat{\mu}(j_{t^*}, \alpha_{t^*}, t^*) = (j_{t^*} + \delta_{t^* \oplus 1}(\alpha_{t^*}) - \delta_{t^*}(\alpha_{t^*}), \alpha_{t^*}, t^* \oplus 1),$$

and the non-boundary dummy agent $(j_{t^*}, \alpha_{t^*}, t^*)$ prefers $(j_{t^*\oplus 1}, \alpha_{t^*\oplus 1}, t^* \oplus 1)$ to $\hat{\mu}(j_{t^*}, \alpha_{t^*}, t^*)$, we have $j_{t^*\oplus 1} < j_{t^*} + \delta_{t^*\oplus 1}(\alpha_{t^*}) - \delta_{t^*}(\alpha_{t^*})$, which contradicts the definition of t^*.

Case 2.2: $j_{t^*} = (k-1)^2$. Then $\delta_{t^*}(\alpha_{t^*}) = 1$ since $j_{t^*} - \delta_{t^*}(\alpha_{t^*}) < (k-1)^2$. So $\alpha_{t^*} \in I \times \{t^*\}$, and hence $\delta_{t^*\oplus 1}(\alpha_{t^*}) = 0$. Since

$$\hat{\mu}(j_{t^*}, \alpha_{t^*}, t^*) = (j_{t^*} - 1, \alpha_{t^*}, t^* \oplus 1)$$

and the boundary dummy agent $(j_{t^*}, \alpha_{t^*}, t^*)$ prefers $(j_{t^*\oplus 1}, \alpha_{t^*\oplus 1}, t^* \oplus 1)$ to $\hat{\mu}(j_{t^*}, \alpha_{t^*}, t^*)$, we have $j_{t^*\oplus 1} < j_{t^*} - 1 = j_{t^*} + \delta_{t^*\oplus 1}(\alpha_{t^*}) - \delta_{t^*}(\alpha_{t^*})$, which contradicts the definition of t^*. □

Proof of Lemma 3. Suppose X has a weakly stable matching μ. Let $\hat{\mu}$ be the matching in $\hat{X}$ induced by μ. It suffices to show that $\hat{\mu}$ does not admit a strongly blocking family.

For the sake of contradiction, suppose $\hat{\mu}$ admits a strongly blocking family

$$((j_0, \alpha_0, 0), \ldots, (j_{k-1}, \alpha_{k-1}, k-1)).$$

Lemma 8 implies that for every $t \in \{0, \ldots, k-1\}$, we have $j_t - \delta_t(\alpha_t) \geq (k-1)^2$. Since $j_t \leq (k-1)^2$ and $\delta_t(\alpha_t) \geq 0$, we deduce that $j_t = (k-1)^2$ and $\delta_t(\alpha_t) = 0$ for every $t \in \{0, \ldots, k-1\}$. Hence for every $t \in \{0, \ldots, k-1\}$, there exists s_t such that $(j_t, \alpha_t, t) = R_t[s_t]$.

Let $t^* \in \{0, \ldots, k-1\}$ such that

$$s_{t^*} = \min_{t \in \{0, \ldots, k-1\}} s_t.$$

Since $\hat{\mu}(R_{t^*}[s_{t^*}]) = R_{t^*\oplus 1}[s_{t^*}]$ and the boundary dummy agent $R_{t^*}[s_{t^*}]$ prefers boundary dummy agent $R_{t^*\oplus 1}[s_{t^*\oplus 1}]$ to boundary dummy agent $\hat{\mu}(R_{t^*}[s_{t^*}])$, we deduce that $R_{t^*\oplus 1}[s_{t^*\oplus 1}]$ is lexicographically smaller than $R_{t^*\oplus 1}[s_{t^*}]$. Hence $s_{t^*\oplus 1} < s_{t^*}$, which contradicts the definition of t^*. □

5 Concluding Remarks

We have shown that a 3-DSM-CYC instance need not admit a weakly stable matching, and that it is NP-complete to determine whether a given 3-DSM-CYC instance admits a weakly stable matching. It seems that for the three-dimensional stable matching problem, none of the preference structures studied in the literature admits a non-trivial generalization of the existence theorem of Gale and Shapley. (The existence result in Danilov's model [5] follows from applying the Gale-Shapley algorithm in a straightforward manner.) It would be interesting to consider solution concepts such as popular matchings instead of stable matchings in the multi-dimensional matching context.

The 3-DSMI-CYC instance with no weakly stable matching presented by Biró and McDermid [2, Lemma 1] has 6 agents of each type. The reduction of Section 4.1 blows up the number of agents by a factor of $k[(k-1)^2+1]$. Thus, for $k = 3$, we obtain an explicit construction of a 3-DSM-CYC instance with no weakly stable matching and $6 \cdot 15 = 90$ agents of each type. It would be interesting to identify smaller 3-DSM-CYC instances with no weakly stable matching.

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