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Algorithms for Stable Matching with Indifferences

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Algorithms for Stable Matching with Indifferences

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Algorithms for Stable Matching with Indifferences

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In the stable matching problem, given a two-sided matching market where each agent has ordinal preferences over the agents on the other side, we would like to find a bipartite matching such that no pair of agents prefer each other to their partners. Indifferences in preferences of the agents arise naturally in large-scale centralized matching schemes. We consider stable matching models where indifferences may occur in the preferences and address some of the related algorithmic challenges.

In the first part of this dissertation, we study group strategyproofness and Pareto-stability in the stable matching market with indifferences. We present Pareto-stable mechanisms that are group strategyproof for one side of the market. Our key technique involves modeling the stable matching market as a generalized assignment game.

In the second part of this dissertation, we study the problem of finding maximum stable matchings when preference lists are incomplete and contain one-sided ties. We present a polynomial algorithm that achieves an approximation ratio of $1 + (1 - \frac{1}{L})^L$, where L is the maximum tie length. Our algorithm is based on a proposal process in which numerical

priorities are adjusted according to the solution of a linear program, and are used for tie-breaking purposes. Our main idea is to use an infinitesimally small step size for incrementing the priorities. Our analysis involves a charging argument and an infinite-dimensional factor-revealing linear program. We also show that the same ratio of $1 + (1 - \frac{1}{L})^L$ is an upper bound on the integrality gap, which matches the known lower bound. For the case of one-sided ties where the maximum tie length is two, our result implies an approximation ratio and integrality gap of $\frac{5}{4}$, which matches the known UG-hardness result.

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Chapter 1

Introduction

1.1 Stable Matching and Indifferences

The stable matching problem involves a market consisting of two disjoint sets of agents. The agents in the two sets are referred to as men and women in the literature. Each agent has preferences over the agents of the opposite sex. The goal is to find a set of disjoint man-woman pairs, called a matching, such that no man and woman prefer each other to their partners. Such matchings are said to be stable.

The stable matching problem was introduced by Gale and Shapley [32], who showed that stable matchings always exist and can be computed by a linear-time algorithm known as the deferred acceptance process. In their paper, preferences are strict, but agents are allowed to declare some other agents as unacceptable. The preference list of an agent is said to contain a tie when the agent is indifferent between two or more agents of the opposite sex. The preference list of an agent is said to be incomplete when one or more agents of the opposite sex are unacceptable to the agent. The stable matching model with incomplete lists and strict preferences is well-studied and known to possess various elegant mathematical properties, including the lattice property [60, attributed to Conway], the lone wolf theorem [71], the characterization of the stable matching polytope [90], and the existence of the one-sided group strategyproof mechanism [23]. Since then, this model and its various generalizations have led to a large body of research in computer science [36, 67] and economics [89].

Indifferences arise naturally in real-world applications. When preferences contain ties, the notion of stability can be generalized to weak stability, strong stability, or super-stability [47]. Weakly stable matchings can be computed by breaking all ties arbitrarily before invoking the Gale-Shapley algorithm. Strongly stable matchings and super-stable matchings do not always exist, but polynomial-time algorithms have been developed to find such solutions when they exist [47]. This dissertation mainly focuses on weakly stable matchings, which always exist.

In this dissertation, we study algorithms for weakly stable matchings that achieve additional desirable properties. In the first part of this dissertation, we present Pareto-stable mechanisms that are group strategyproof for one side of the market, where preferences may be incomplete and contain ties. In the second part of this dissertation, we study polynomial-time approximation algorithms for finding maximum weakly stable matchings, where preferences on one side of the market may be incomplete and contain ties.

Other problems involving weak stability have been considered in the literature, but many of them have strong hardness results. The problem of finding a minimum regret weakly stable matching or an egalitarian weakly stable matching is hard to approximate, even if ties only occur on one side of the market [37, 68]. It is also hard to approximate the problem of finding a sex-equal weakly stable matching [37].

1.2 Generalizations of the Stable Matching Model

The stable matching model has been generalized in various ways to allow agents to have more expressive preferences. Many results for the stable matching model with strict preferences can be extended to these models.

1.2.1 The College Admissions Problem

The college admissions problem is a many-to-one matching market introduced by Gale and Shapley [32]. The agents in the college admissions model are called students and colleges, and each agent has preferences over the agents on the other side. In addition, each college also has a capacity value which represents the maximum number of students that can be matched with the college, and outcomes of a mechanism are capacitated matchings. The preferences of a college over individual students can be extended to preferences over groups of students using the notion of responsiveness [84]. Stable matchings exist, and can be computed by reducing the college admission problem to the stable matching problem [34]. When preferences are strict and responsive, the set of stable matchings forms a lattice [89], and the student-optimal mechanism is group strategyproof for the students [23]. When the roles of the students and colleges are replaced by residents and hospitals, respectively, this model is also called the hospitals/residents problem [36]. The rural hospital theorem [85] for the hospitals/residents problem is a generalization of the lone wolf theorem.

1.2.2 Matching with Contracts

The model of matching with contracts is a generalization of the college admissions problem introduced by Hatfield and Milgrom [43]. It involves a many-to-one matching market where there can be multiple contract terms for matching a given pair of agents. Each agent has strict preferences over the contracts involving the agent. When preferences satisfy substitutability and the law of aggregate demand, the stable matchings form a non-empty lattice and satisfy the rural hospital theorem [43]. A group strategyproof stable mechanism for one side of the market also exists under the same assumptions [41].

1.2.3 Matching in Supply Chain Networks

The model of matching in supply chain networks is a further generalization introduced by Ostrovsky [74]. It involves a market where agents in an acyclic network negotiate bilateral contracts. Each agent has strict preferences over the bilateral contracts involving the agent. When preferences satisfy full substitutability and the laws of aggregate demand and supply, the stable matchings form a non-empty lattice and satisfy a generalized form of the rural hospital theorem [42]. A group strategyproof stable mechanism for the agents with unit demand also exists under the same assumptions [42]. This model subsumes the many-to-many matching model.

1.3 Related Two-Sided Matching Problems

The stable matching model is known to share many common properties with other related two-sided matching problems such as the housing market problem and the asssignment game.

1.3.1 The Housing Market Problem

Like the stable matching problem, the housing market problem introduced by Shapley and Scarf [91] is a two-sided matching model with ordinal preferences. It involves a set of agents, each of which owns a house. Each agent has preferences over the houses. Perfect matchings between the agents and the houses represent exchanges of houses among the agents without any monetary transfer. A suitable solution concept is the core. Core allocations always exist and can be computed efficiently by the top trading cycles algorithm [91, attributed to Gale], which corresponds to a group strategyproof mechanism [7]. When indifferences are allowed, there exist strategyproof mechanisms that produce Pareto-optimal core allocations in polynomial time [54, 77].

1.3.2 The Assignment Game

A simple two-sided market with monetary transfers is the assignment game introduced by Shapley and Shubik [92]. It involves a set of unit-demand buyers and a set of sellers. Each seller has an item to sell, and each buyer has a valuation for each item. An outcome consists of a matching between the buyers and the sellers along with a price for each item. The utility of a buyer is assumed to be the difference between their valuation for their assigned item and its price. An outcome is envy-free if every buyer is assigned an item that maximizes their utility. Shapley and Shubik showed that finding an envy-free matching corresponds to solving the weighted bipartite matching problem, which in turns corresponds to linear optimization on the matching polytope. They also showed that envy-free utility payoffs correspond to feasible solutions in the dual linear program and form a non-empty lattice. Demange et al. [17] described an ascending auction mechanism that produces buyer-optimal outcomes. This mechanism is strategyproof [65], and is equivalent to the Vickrey-Clarke-Groves mechansim [12, 35, 96]. The assignment game has been generalized to accommodate a larger class of utility functions [13, 16, 56], and to the exchange model [31, 79].

To find an envy-free solution for the assignment game, the Hungarian method [62] can be used to compute a maximum-weight bipartite matching. More efficient algorithms are known for special cases such as sparse edge-weighted graphs [30], graphs with small integer edge weights [22], vertex-weighted convex bipartite graphs [76], vertex-weighted two-directional orthogonal ray graphs [78], and graphs with linear edge weights [19, 21].

1.4 Multi-Sided Systems

The problem of generalizing the stable matching problem to three-dimensional matchings in three-sided systems was posed by Knuth [60]. Various restrictions on the preferences of the agents have been considered. When each agent has preferences over pairs of agents from the other two sides, it is NP-complete to decide whether a stable matching exists [73, 94], even if the preferences are consistent with product orders [44]. When the agents have lexicographically acyclic preferences over pairs of agents from the other two sides, Danilov [14] showed that multi-sided stable matchings alway exist and can be obtained efficiently by computing two-sided stable matchings in a natural hierarchical manner. When the agents have purely cyclic preferences over agents of another side, it is NP-complete to decide whether a stable matching exists [8, 64].

1.5 Applications

The stable matching problem has applications in centralized matching schemes. One classical example is the National Resident Matching Program, which recruits graduating physicians to residency programs. Previously known as the National Intern Matching Program, it was shown to be equivalent to the deferred acceptance process by Roth [82], who credited its success to the stability of its output. According to a more recent report by Roth and Peranson [87], the program underwent a redesign and is matching approximately twenty thousand jobs annually. In the United Kingdom, the Scottish Foundation Allocation Scheme is a similar mechanism for matching medical students to hospitals; in this mechanism, ties may appear at the end of the preference lists of the hospitals [68].

The stable matching problem also has applications in school choice [3]. In this setting, we have a set of students and a set of schools. Each student has preferences over the schools, and each school has a capacity value and a priority order over the students. The key difference between the priority orders of the schools in the school choice setting and the preferences of the colleges in the college admissions problem is that the former may be determined by local policies instead of true preferences. The counterpart of stability is fairness. A matching is unfair if there exists a student and a school such that the student prefers the school to their match and has a higher priority at the school than some other student matched to the school. Two strategyproof mechanisms suggested by Abdulkadiroğlu and Sönmez [3] for the school choice problem are the student-proposing deferred acceptance process and the top trading cycles algorithm. The former is fair and the latter is Pareto-optimal for the students. However, no mechanism can be both fair and Pareto-optimal for the students [81]. In New York City high school admissions, over ninety thousand students are assigned to high schools every year using the deferred acceptance process [1]. A feature of the priority orders of those high schools is the presence of ties, and field data was used by Abdulkadiroğlu et al. [2] to analyze the welfare of the students under different methods for handling indifferences. Other practical considerations of school choice that have been studied include diversity [25, 26, 61] and community cohesion [5].

Chapter 2

The Two-Sided Matching Models

2.1 Stable Matching Market

The stable matching market involves a set I of men and a set J of women. The sets I and J are assumed to be disjoint and finite. Furthermore, we assume that the sets I and J do not contain the element 0, which we use to denote being unmatched. The preference relation of each man $i \in I$ is specified by a binary relation $\geq_i$ over $J \cup \{0\}$ that satisfies transitivity[1] and totality[2]. Similarly, the preference relation of each woman $j \in J$ is specified by a binary relation $\geq_j$ over $I \cup \{0\}$ that satisfies transitivity and totality. We denote this stable matching market as $(I, J, \{\geq_i\}_{i \in I}, \{\geq_j\}_{j \in J})$.

The preference relations in the stable matching market need not satisfy antisymmetry[3] in general. For each agent $k \in I \cup J$, we use $>_k$ and $=_k$ to denote the asymmetric part[4] and the symmetric part[5] of $\geq_k$ respectively. We say that agent k *weakly prefers* agent k' to

[1]A binary relation $\succeq$ over a set K is said to satisfy *transitivity* if for every $k_1, k_2, k_3 \in K$ such that $k_1 \succeq k_2$ and $k_2 \succeq k_3$, we have $k_1 \succeq k_3$.

[2]A binary relation $\succeq$ over a set K is said to satisfy *totality* if for every $k_1, k_2 \in K$, we have either $k_1 \succeq k_2$ or $k_2 \succeq k_1$.

[3]A binary relation $\succeq$ over a set K is said to satisfy *antisymmetry* if for every $k_1, k_2 \in K$ such that $k_1 \succeq k_2$ and $k_2 \succeq k_1$, we have $k_1 = k_2$.

[4]The *asymmetric part* of a binary relation $\succeq$ over a set K is the binary relation $\succ$ over the set K such that for every $k_1, k_2 \in K$, we have $k_1 \succ k_2$ if and only if $k_1 \succeq k_2$ and $\neg(k_2 \succeq k_1)$.

[5]The *symmetric part* of a binary relation $\succeq$ over a set K is the binary relation $\sim$ over the set K such that for every $k_1, k_2 \in K$, we have $k_1 \sim k_2$ if and only if $k_1 \succeq k_2$ and $k_2 \succeq k_1$.

agent k'' if $k' \geq_k k''$. If agent k weakly prefers agent k' to agent k'', then either $k' >_k k''$ or $k' =_k k''$. We say that agent k *prefers* agent k' to agent k'' in the former case, and is *indifferent* between agent k' and agent k'' in the latter case. A *tie* in the preference list of an agent k is an equivalence class of size at least 2 with respect to the equivalence relation $=_k$. If the preference list of agent k has no tie, we say that agent k has *strict preferences*. Otherwise, agent k is said to have *weak preferences*.

Acceptability can be defined in terms of the preference relations. For every agent k and k' of opposite sexes, either $k' \geq_k 0$ or $0 >_k k'$. We say that agent k' is *acceptable* to agent k in the former case, and *unacceptable* to agent k in the latter case. If man i and woman j are acceptable to each other, we say that (i, j) is an *acceptable pair*. Otherwise, (i, j) is an *unacceptable pair*. Agent k is said to have a *complete preference list* if every agent k' of the opposite sex is acceptable to agent k. Otherwise, agent k is said to have an *incomplete preference list*.

A matching is a subset $\mu \in I \times J$ such that for every $(i, j), (i', j') \in \mu$, we have $i = i'$ if and only if $j = j'$. We denote

$$\mu(i) = \begin{cases} j & \text{if } (i, j) \in \mu \\ 0 & \text{if } (i, j) \notin \mu \text{ for every woman } j \in J \end{cases}$$

for every man $i \in I$ and

$$\mu(j) = \begin{cases} i & \text{if } (i, j) \in \mu \\ 0 & \text{if } (i, j) \notin \mu \text{ for every man } i \in I \end{cases}$$

for every woman $j \in J$. For every agent k, we say that agent k is *matched* in μ if $\mu(k) \neq 0$. Otherwise, agent k is said to be *unmatched* in μ.

Weak stability can be defined in terms of individual rationality and strongly blocking pairs. A matching μ is *individually rational* if $\mu(k) \geq_k 0$ for every agent $k \in I \cup J$. A man-woman pair $(i, j) \in I \times J$ is a *strongly blocking pair* of a matching μ if $j >_i \mu(i)$ and $i >_j \mu(j)$. A matching μ is *weakly stable* if the matching μ is individually rational and admits no strongly blocking pair.

Given a set I of men and a set J of women, a mechanism is an algorithm that takes the preference profiles $\{\geq_i\}_{i \in I}$ and $\{\geq_j\}_{j \in J}$ as input, and produces a matching μ as output.

2.2 Generalized Assignment Game

The generalized assignment game studied by Demange and Gale [16] involves two disjoint finite sets I and J of agents, which we call men and women respectively. We assume that the sets I and J do not contain the element 0, which we use to denote being unmatched. For each man $i \in I$ and woman $j \in J$, the compensation function $f_{i,j}(u_i)$ represents the compensation that man i needs to receive in order to attain utility u_i when he is matched to woman j. Similarly, for each man $i \in I$ and woman $j \in J$, the compensation function $g_{i,j}(v_j)$ represents the compensation that woman j needs to receive in order to attain utility v_j when she is matched to man i. The compensation functions $f_{i,j}$ and $g_{i,j}$ are assumed to be increasing and invertible for every man $i \in I$ and woman $j \in J$. Moreover, each man $i \in I$ has a reserve utility r_i and each woman $j \in J$ has a reserve utility s_j. We denote this generalized assignment game as $(I, J, \mathbf{f}, \mathbf{g}, \mathbf{r}, \mathbf{s})$, where $\mathbf{f} = \{f_{i,j}\}_{(i,j) \in I \times J}$ and $\mathbf{g} = \{g_{i,j}\}_{(i,j) \in I \times J}$ are the compensation functions, and $\mathbf{r} = \{r_i\}_{i \in I}$ and $\mathbf{s} = \{s_j\}_{j \in J}$ are the reserve utilities.

An *outcome* is a triple $(\mu, \mathbf{u}, \mathbf{v})$, where $\mu \subseteq I \times J$ is a matching, $\mathbf{u} = \{u_i\}_{i \in I} \in \mathbb{R}^I$ is the utility vector of the men, and $\mathbf{v} = \{v_j\}_{j \in J} \in \mathbb{R}^J$ is the utility vector of the women. An outcome $(\mu, \mathbf{u}, \mathbf{v})$ is *feasible* if the following conditions hold.

(1) For every $(i, j) \in \mu$, we have $f_{i,j}(u_i) + g_{i,j}(v_j) \leq 0$.

(2) For every $i \in I$ such that $\mu(i) = 0$, we have $u_i = r_i$.

(3) For every $j \in J$ such that $\mu(j) = 0$, we have $v_j = s_j$.

A feasible outcome $(\mu, \mathbf{u}, \mathbf{v})$ is *individually rational* if $u_i \geq r_i$ and $v_j \geq s_j$ for every man $i \in I$ and woman $j \in J$. An individually rational outcome $(\mu, \mathbf{u}, \mathbf{v})$ is *stable* if $f_{i,j}(u_i) + g_{i,j}(v_j) \geq 0$ for every man $i \in I$ and woman $j \in J$.

A stable outcome $(\mu, \mathbf{u}, \mathbf{v})$ is *man-optimal* if for any stable outcome $(\mu', \mathbf{u}', \mathbf{v}')$ we have $u_i \geq u'_i$ for every man $i \in I$. It is known that man-optimal outcomes always exist [16, Property 2] and satisfy the following useful properties.

Lemma 2.1 ([16, Lemma 4])**.** *Let $(\mu, \mathbf{u}, \mathbf{v})$ be a man-optimal outcome of generalized assignment game $(I, J, \mathbf{f}, \mathbf{g}, \mathbf{r}, \mathbf{s})$. Let $J' \subseteq J$ be a non-empty subset of woman such that $v_j \neq s_j$ for every $j \in J'$. Then there exists a man $i' \in I$ and a woman $j' \in J$ such that $\mu(i') \notin J'$ and $f_{i',j'}(u_{i'}) + g_{i',j'}(v_{j'}) = 0$.*

Theorem 2.2 ([16, Theorem 2])**.** *Let $(\mu, \mathbf{u}, \mathbf{v})$ be a man-optimal outcome of generalized assignment game $(I, J, \mathbf{f}, \mathbf{g}, \mathbf{r}, \mathbf{s})$, and $(\mu', \mathbf{u}', \mathbf{v}')$ be a stable outcome of generalized assignment game $(I, J, \mathbf{f}', \mathbf{g}, \mathbf{r}', \mathbf{s})$. Let $I' \subseteq I$ be a non-empty subset such that for every man $i \in I \setminus I'$ and woman $j \in J$, we have $f_{i,j} = f'_{i,j}$ and $r_i = r'_i$. Then there exists a man $i' \in I'$ such that $\mu'(i') \neq 0$ implies $f_{i',\mu'(i')}(u_{i'}) \geq f'_{i',\mu'(i')}(u'_{i'})$.*

Chapter 3

Group Strategyproof Pareto-Stable Mechanisms

In this chapter, we study group strategyproofness and Pareto-stability in the stable matching market with indifferences. Our main result is the existence of a group strategyproof Pareto-stable mechanism. We achieve this by modeling the stable matching market as an appropriate form of the generalized assignment game. This result appears in our conference paper [20]. We omit from this dissertation the details of the equivalence of our mechanism and an efficient implementation using iterated unit-demand auctions, which can be found in the dissertation [18] of one of the co-authors of the paper.

In Section 3.1, we review the prior work related to this problem. In Section 3.2, we present an overview of our techniques. In Section 3.3, we define the notion of group strategyproofness and Pareto-stability formally. In Section 3.4, we present a reduction from the stable matching market to the associated generalized assignment game. In Section 3.5, we show that our mechanism is Pareto-stable. In Section 3.6, we show that our mechanism is group strategyproof for the men.

3.1 Related Work

In Section 3.1.1, we review the prior work on strategic issues related to stable matchings. In Section 3.1.2, we review the prior work on algorithms for computing Pareto-stable matchings.

3.1.1 Strategyproofness

Strategic issues are well-understood for the stable matching model with strict preferences. Roth [81] showed that strategyproofness for all agents and Pareto-optimality can be obtained by serial dictatorship, but such a mechanism does not produce stable matchings in general. In fact, it is not hard to see that serial dictatorship is also group strategyproof. He also showed that the Gale-Shapley algorithm, which produces the unique man-optimal stable matching when preferences are strict, is strategyproof for the men. Dubins and Freedman [23] showed the stronger result that the same mechanism is actually group strategyproof for the men. Alternative proofs of this result can be obtained using the blocking lemma by Hwang [34] or the linear programming formulation [88]. However, Gale and Sotomayor [33] showed that this mechanism is not strategyproof for the women. Their proof can be adapted to show that when preferences are strict, a stable matching mechanism is strategyproof for the men only if it produces the man-optimal stable matching, and hence is equivalent to the Gale-Shapley algorithm. Since it is not strategyproof for the women, the women may benefit by expressing preference profiles that are different from their true preferences. It is known that when preferences are strict and the men state their true preferences, every output of the mechanism at a strategic equilibrium[1] is stable [83] and every stable matching is an output of the mechanism at some strategic equilibrium [33]. In particular, the women-optimal stable

[1]An expressed preference profile is a strategic equilibrium if no agent can benefit by unilaterally changing their expressed preferences.

matching corresponds to a strong equilibrium for the women[2] when preferences are strict, even though it may not be the only strong equilibrium for the women [33]. It is also shown that the set of non-dominated strategies of the women has a nice characterization in the case of strict preferences [33, 81]. These strategic issues have also been studied in a model that does not assume complete information [86].

3.1.2 Pareto-Stability

The notion of Pareto-stability was coined by Sotomayor [93], who argued that it is an appropriate solution concept for the stable matching market with indifferences. Erdil and Ergin [28] showed that weakly stable matchings obtained by breaking ties arbitrarily may not be Pareto-optimal. They present an $O(n^4)$-time algorithm that obtains a Pareto-stable matching by applying successive Pareto-improvements to a weakly stable matching, where n is the total number of men and women. Pareto-stable matchings also exist and can be computed in strongly polynomial time for many-to-many matchings [9] and multi-unit matchings [10]. Instead of relying on the characterization of Pareto-improvement chains and cycles, Kamiyama [55] gave another efficient algorithm for Pareto-stable many-to-many matchings based on rank-maximal matchings. However, none of these mechanisms address strategyproofness.

[2] A strategic equilibrium is a strong equilibrium for the women if no coalition of women can all strictly benefit by changing their preferences.

3.2 Overview of the Techniques

The key technique for the design of our group strategyproof Pareto-stable mechanism involves reducing the stable matching market with indifferences to the generalized assignment game.

Demange and Gale [16] established various properties of the generalized assignment game including the lattice property and the existence of man-optimal outcomes. These properties are known to hold for the stable matching market in the case of strict preferences [60, attributed to Conway], but fail in the case of weak preferences [82]. Given the similarities between stable matching markets and generalized assignment games, it is natural to ask whether stable matching markets can be modeled as generalized assignment games. Demange and Gale discuss this question and state that “the model of [Gale and Shapley] is not a special case of our model”. The basic obstacle is that it is unclear how to model an agent’s preferences within the framework of a generalized assignment game. On the one hand, even though ordinal preferences can be converted into cardinal utility values, such preferences are expressed in a manner that is independent of any monetary transfer. On the other hand, the framework demands that there is an amount of money that makes an agent indifferent between any two agents on the other side of the market.

Our approach converts a stable matching market with indifferences into an associated generalized assignment game. While these are both two-sided markets that involve the same set of agents, the utilities achieved under an outcome in the associated generalized assignment game may not be equal to the utilities under a corresponding solution in the stable matching market. Nevertheless, we are able to establish useful relationships between certain sets of solutions for these two markets.

Our first such result, Theorem 3.5, shows that Pareto-stability in the stable matching market with indifferences follows from stability in the associated generalized assignment game, even though it does not follow from weak stability in the stable matching market with indifferences. This can be seen as a partial analogue to the case of strict preferences, in which stability in the stable matching market implies Pareto-stability [32]. This also demonstrates that, in addition to using the deferred acceptance procedure to solve the generalized assignment game [13], we can use the generalized assignment game to solve the stable matching problem with indifferences.

In Lemma 3.6, we establish that the utility achieved by any man in a man-optimal solution to the associated generalized assignment game uniquely determines the tier of preference to which that man is matched in the stable matching market with indifferences. Another consequence of this lemma is that any matched man in a man-optimal outcome of the associated generalized assignment game receives at least one unit of money from his partner. We can then deduce that if a man strictly prefers his partner to a woman, then the woman has to offer a large amount of money in order for the man to be indifferent between her offer and that of his partner. Since individual rationality prevents any woman from offering such a large amount of money, this explains how we overcome the obstacle of any man being matched with a less preferred woman in exchange for a sufficiently large payment.

A key result established by Demange and Gale is that the man-optimal mechanism is group strategyproof for the men. Using this result and Lemma 3.6, we are able to show in Theorem 3.7 that group strategyproofness for the men in the stable matching market with indifferences is achieved by man-optimality in the associated generalized assignment game, even though it is incompatible with man-Pareto-optimality in the stable matching market with indifferences [27, 57]. This can be seen as a partial analogue to the case of strict preferences, in which man-optimality implies group strategyproofness [23].

3.3 Preliminary Definitions

Consider a set I of men and a set J of women. A mechanism is *group strategyproof for the men* if for every preference profile $\{\geq_i\}_{i \in I}$, $\{\geq'_i\}_{i \in I}$, and $\{\geq_j\}_{j \in J}$, either the preference profiles $\{\geq_i\}_{i \in I}$ and $\{\geq'_i\}_{i \in I}$ are identical or there exists a man $i' \in I$ with preference relation $\geq'_{i'}$ different from $\geq_{i'}$ such that $\mu(i') \geq_{i'} \mu'(i')$, where μ and μ' are the matchings produced by the mechanism given $(\{\geq_i\}_{i \in I}, \{\geq_j\}_{j \in J})$ and $(\{\geq'_i\}_{i \in I}, \{\geq_j\}_{j \in J})$ respectively. (Such a man i' belongs to the coalition but is not matched to a strictly preferred woman by expressing the preference relation $\geq'_{i'}$ instead of his true preference relation $\geq_{i'}$.)

We remark that this notion of group strategyproofness is different from strong group strategyproofness, in which at least one man in the coalition gets matched to a strictly preferred partner while the other men in the coalition get matched to weakly preferred partners. It is known that strong group strategyproofness for the men is impossible for the stable matching market even when preferences are strict [23, attributed to Gale]. This notion of group strategyproofness also assumes no side payments within the coalition of men. The impossibility of strong group strategyproofness for the men implies the impossibility of group strategyproofness for the men when side payments are allowed.

Consider a stable matching market $(I, J, \{\geq_i\}_{i \in I}, \{\geq_j\}_{j \in J})$. For any matchings μ and μ', we say that the binary relation $\mu \geq_{I \cup J} \mu'$ holds if $\mu(k) \geq_k \mu'(k)$ for every agent $k \in I \cup J$. We use $>_{I \cup J}$ to denote the asymmetric part of $\geq_{I \cup J}$. A matching μ is said to *Pareto-dominate* a matching μ' if $\mu >_{I \cup J} \mu'$ holds. A matching μ is said to be *Pareto-optimal* if the matching μ is not Pareto-dominated by any matching. A matching μ is said to be *Pareto-stable* if the matching μ is weakly stable and Pareto-optimal.

We remark that the notion of Pareto-optimality here is different from Pareto-optimality for the men, which only takes into account the preferences of the men. It is known that man-Pareto-optimality is not compatible with strategyproofness for the stable matching market with indifferences [27, 57]. The notion of Pareto-optimality here is also different from Pareto-optimality in expected utility, which permits Pareto-domination by non-pure outcomes. A result of Zhou [98] implies that Pareto-optimality in expected utility is not compatible with strategyproofness for the stable matching market with indifferences.

3.4 The Associated Generalized Assignment Game

We construct the generalized assignment game $(I, J, \mathbf{f}, \mathbf{g}, \mathbf{r}, \mathbf{s})$ associated with stable matching market $(I, J, \{\geq_i\}_{i \in I}, \{\geq_j\}_{j \in J})$ as follows. We take $N \geq |I| + 1$ and define $\boldsymbol{\pi} = \{\pi_i\}_{i \in I}$ such that each man $i \in I$ is associated with a fixed and distinct priority $\pi_i \in \{1, 2, \ldots, N-1\}$. We convert the preference relations $\{\geq_i\}_{i \in I}$ of the men to integer-valued utility values $\mathbf{a} = \{a_{i,j}\}_{(i,j) \in I \times (J \cup \{0\})}$ such that for every man $i \in I$ and woman $j', j'' \in J \cup \{0\}$, we have $j' \geq_i j''$ if and only if $a_{i,j'} \geq a_{j,j''}$. Similarly, we convert the preference relations $\{\geq_j\}_{j \in J}$ of the women to integer-valued utility values $\mathbf{b} = \{b_{i,j}\}_{(i,j) \in (I \cup \{0\}) \times J}$ such that for every woman $j \in J$ and man $i', i'' \in I \cup \{0\}$, we have $i' \geq_j i''$ if and only if $b_{i',j} \geq b_{i'',j} \geq 1$. Also, we take

$$\lambda = \max_{(i,j) \in (I \cup \{0\}) \times J} (b_{i,j} + 1)N.$$

We define the compensation functions $\mathbf{f} = \{f_{i,j}\}_{(i,j)\in I\times J}$ and $\mathbf{g} = \{g_{i,j}\}_{(i,j)\in I\times J}$ such that

$$f_{i,j}(u_i) = u_i \lambda^{-a_{i,j}}$$

and

$$g_{i,j}(v_j) = v_j - (b_{i,j}N + \pi_i)$$

for every man $i \in I$ and woman $j \in J$. We define the reserve utilities $\mathbf{r} = \{r_i\}_{i\in I}$ and $\mathbf{s} = \{s_j\}_{j\in J}$ such that

$$r_i = \pi_i \lambda^{a_{i,0}}$$

and

$$s_j = b_{0,j}N$$

for every man $i \in I$ and woman $j \in J$. We denote this associated generalized assignment game as $(I, J, \boldsymbol{\pi}, N, \lambda, \mathbf{a}, \mathbf{b})$. For better readability, we write $\exp_\lambda(\xi)$ to denote λ^ξ.

In order to achieve group strategyproofness, we require that $\boldsymbol{\pi}$ and N do not depend on the preferences $\{\geq_i\}_{i\in I}$ of the men. We further require that $\{b_{i,j}\}_{(i,j)\in(I\cup\{0\})\times J}$ does not depend on the preferences $\{\geq_i\}_{i\in I}$ of the men, and that $\{a_{i',j}\}_{j\in J\cup\{0\}}$ does not depend on the other preferences $\{\geq_i\}_{i\in I\setminus\{i'\}}$ for any man $i' \in I$. In other words, a man $i' \in I$ is only able to manipulate his own utilities $\{a_{i',j}\}_{j\in J\cup\{0\}}$. One way to satisfy these conditions is to define $a_{i',j'}$ as the number of women $j \in J \cup \{0\}$ such that $j' \geq_{i'} j$ for every man $i' \in I$ and woman $j' \in J \cup \{0\}$, and to define $b_{i',j'}$ as the number of men $i \in I \cup \{0\}$ such that $i' \geq_{j'} i$ for every man $i' \in I \cup \{0\}$ and woman $j' \in J$.

Intuitively, each woman has a compensation function with the same form as a buyer in the assignment game [92]. The valuation $(b_{i,j}N + \pi_i)$ that woman j assigns to man i has a first-order dependence on her preferences over the men and a second-order dependence on the priorities of the men, which are used to break any ties in her preferences. From the perspective of man i, if he highly prefers a woman j, he assigns a large exponent $a_{i,j}$ in the slope of his compensation function associated with woman j, and thus expects only a small amount of compensation if he is matched to woman j.

3.5 Pareto-Stability

In this section, we study the Pareto-stability of matchings in the stable matching market that correspond to stable outcomes in the associated generalized assignment game. First, in Lemmas 3.1 and 3.2, we show that individual rationality in the associated generalized assignment game implies individual rationality in the stable matching market. Then, in Lemma 3.3, we show that stability in the associated generalized assignment game implies weak stability in the stable matching market. Finally, in Lemma 3.4 and Theorem 3.5, we show that stability in the associated generalized assignment game is sufficient for Pareto-stability in the stable matching market.

Lemma 3.1. *Let $(\mu, \mathbf{u}, \mathbf{v})$ be an individually rational outcome in an associated generalized assignment game $(I, J, \boldsymbol{\pi}, N, \lambda, \mathbf{a}, \mathbf{b})$. Let $i \in I$ be a man and $j \in J$ be a woman. Then*

$$0 < \exp_\lambda(a_{i,0}) \leq u_i < \exp_\lambda(a_{i,\mu(i)} + 1) \quad \textit{and} \quad 0 \leq b_{0,j}N \leq v_j < (b_{\mu(j),j} + 1)N.$$

Proof. The lower bounds

$$u_i \geq \pi_i \exp_\lambda(a_{i,0}) \geq \exp_\lambda(a_{i,0}) > 0 \qquad \text{and} \qquad v_j \geq b_{0,j}N \geq 0$$

follow directly from the individual rationality of $(\mu, \mathbf{u}, \mathbf{v})$. If $\mu(i) = 0$, then the feasibility of $(\mu, \mathbf{u}, \mathbf{v})$ implies $u_i = \pi_i \exp_\lambda(a_{i,0}) < \exp_\lambda(a_{i,0} + 1)$. If $\mu(j) = 0$, then the feasibility of

$(\mu, \mathbf{u}, \mathbf{v})$ implies $v_j = b_{0,j}N < (b_{0,j} + 1)N$. It remains to show that the upper bounds hold when $\mu(i) \neq 0$ and $\mu(j) \neq 0$. Without loss of generality, we may assume that $\mu(i) = j$. So the feasibility of $(\mu, \mathbf{u}, \mathbf{v})$ implies

$$u_i \exp_\lambda(-a_{i,j}) - (b_{i,j}N + \pi_i - v_j) \leq 0.$$

Since $u_i \geq 0$ and $v_j \geq 0$, we have

$$u_i \exp_\lambda(-a_{i,j}) - (b_{i,j}N + \pi_i) \leq 0 \qquad \text{and} \qquad -(b_{i,j}N + \pi_i - v_j) \leq 0.$$

Since $\pi_i < N$ and $b_{i,j}N + \pi_i < \lambda$, we have

$$u_i \exp_\lambda(-a_{i,j}) - \lambda < 0 \qquad \text{and} \qquad -(b_{i,j}N + N - v_j) < 0.$$

Thus $u_i < \exp_\lambda(a_{i,\mu(i)} + 1)$ and $v_j < (b_{\mu(j),j} + 1)N$. □

Lemma 3.2 (Individual Rationality)**.** *Let $(\mu, \mathbf{u}, \mathbf{v})$ be an individually rational outcome in the generalized assignment game $(I, J, \boldsymbol{\pi}, N, \lambda, \mathbf{a}, \mathbf{b})$ associated with stable matching market $(I, J, \{\geq_i\}_{i\in I}, \{\geq_j\}_{j\in J})$. Then μ is an individually rational matching in the stable matching market.*

Proof. Let $i \in I$ be a man and $j \in J$ be a woman. Then, by Lemma 3.1, we have

$$\exp_\lambda(a_{i,0}) < \exp_\lambda(a_{i,\mu(i)} + 1) \qquad \text{and} \qquad b_{0,j}N < (b_{\mu(j),j} + 1)N.$$

Thus $a_{i,\mu(i)} + 1 > a_{i,0}$ and $b_{\mu(j),j} + 1 > b_{0,j}$, and hence $a_{i,\mu(i)} \geq a_{i,0}$ and $b_{\mu(j),j} \geq b_{0,j}$. We conclude that $\mu(i) \geq_i 0$ and $\mu(j) \geq_j 0$. □

Lemma 3.3 (Stability)**.** *Let $(\mu, \mathbf{u}, \mathbf{v})$ be a stable outcome in the generalized assignment game $(I, J, \boldsymbol{\pi}, N, \lambda, \mathbf{a}, \mathbf{b})$ associated with stable matching market $(I, J, \{\geq_i\}_{i\in I}, \{\geq_j\}_{j\in J})$. Then μ is a weakly stable matching in the stable matching market.*

Proof. Since the outcome $(\mu, \mathbf{u}, \mathbf{v})$ is individually rational in the associated generalized assignment game, Lemma 3.2 implies that the matching μ is individually rational in the stable matching market. It remains to show that there is no strongly blocking pair.

For the sake of contradiction, suppose there exists a man $i \in I$ and a woman $j \in J$ such that $j >_i \mu(i)$ and $i >_j \mu(j)$. Then $a_{i,j} \geq a_{i,\mu(i)} + 1$ and $b_{i,j} \geq b_{\mu(j),j} + 1$. Since $(\mu, \mathbf{u}, \mathbf{v})$ is a stable outcome in the associated generalized assignment game, we have

$$\begin{aligned} 0 &\leq u_i \exp_\lambda(-a_{i,j}) - (b_{i,j}N + \pi_i - v_j) \\ &< \exp_\lambda(a_{i,\mu(i)} + 1) \exp_\lambda(-a_{i,j}) - (b_{i,j}N + \pi_i - (b_{\mu(j),j} + 1)N) \\ &\leq 1 - \pi_i, \end{aligned}$$

where the second inequality follows from Lemma 3.1. Thus, $\pi_i < 1$, a contradiction. $\square$

Lemma 3.4. *Let $(\mu, \mathbf{u}, \mathbf{v})$ be a stable outcome in an associated generalized assignment game $(I, J, \boldsymbol{\pi}, N, \lambda, \mathbf{a}, \mathbf{b})$. Let μ' be an arbitrary matching. Then*

$$\sum_{i \in I} \Big(u_i \exp_\lambda(-a_{i,\mu'(i)}) - \pi_i \Big) \geq \sum_{j \in J} \Big(b_{\mu'(j),j} N - v_j \Big).$$

Furthermore, the inequality is tight if and only if the outcome $(\mu', \mathbf{u}, \mathbf{v})$ is stable.

Proof. Since $(\mu, \mathbf{u}, \mathbf{v})$ is a stable outcome in the associated generalized assignment game, the following conditions hold.

(1) $u_i \geq \pi_i \exp_\lambda(a_{i,0})$ for every man $i \in I$ such that $\mu'(i) = 0$.

(2) $u_i \exp_\lambda(-a_{i,\mu'(i)}) + v_{\mu'(i)} - (b_{i,\mu'(i)}N + \pi_i) \geq 0$ for every man $i \in I$ such that $\mu'(i) \neq 0$.

(3) $v_j \geq b_{0,j}N$ for every woman $j \in J$ such that $\mu'(j) = 0$.

Hence, we have

$$
\begin{aligned}
&\sum_{i\in I}\Big(u_i \exp_\lambda(-a_{i,\mu'(i)}) - \pi_i\Big)\\
&= \sum_{\substack{i\in I\\ \mu'(i)\neq 0}} \Big(u_i \exp_\lambda(-a_{i,\mu'(i)}) - \pi_i\Big) + \sum_{\substack{i\in I\\ \mu'(i)= 0}} \Big(u_i \exp_\lambda(-a_{i,0}) - \pi_i\Big)\\
&\geq \sum_{\substack{i\in I\\ \mu'(i)\neq 0}} \Big(u_i \exp_\lambda(-a_{i,\mu'(i)}) - \pi_i\Big)\\
&\geq \sum_{\substack{i\in I\\ \mu'(i)\neq 0}} \Big(b_{i,\mu'(i)}N - v_{\mu'(i)}\Big)\\
&= \sum_{\substack{j\in J\\ \mu'(j)\neq 0}} \Big(b_{\mu'(j),j}N - v_j\Big)\\
&= \sum_{j\in J} \Big(b_{\mu'(j),j}N - v_j\Big) - \sum_{\substack{j\in J\\ \mu'(j)= 0}} \Big(b_{0,j}N - v_j\Big)\\
&\geq \sum_{j\in J} \Big(b_{\mu'(j),j}N - v_j\Big),
\end{aligned}
$$

where the three inequalities follow from conditions (1), (2), and (3), respectively.

Furthermore, if the outcome $(\mu', \mathbf{u}, \mathbf{v})$ is stable, then conditions (1), (2), and (3) are all tight. Hence, the inequality in the lemma statement is also tight.

Conversely, if the inequality in the lemma statement is tight, then conditions (1), (2), and (3) are all tight. Hence, the outcome $(\mu', \mathbf{u}, \mathbf{v})$ is feasible. So, the stability of outcome $(\mu', \mathbf{u}, \mathbf{v})$ follows from the stability of outcome $(\mu, \mathbf{u}, \mathbf{v})$. $\square$

Theorem 3.5 (Pareto-stability)**.** *Let* $(\mu, \mathbf{u}, \mathbf{v})$ *be a stable outcome in the generalized assignment game* $(I, J, \boldsymbol{\pi}, N, \lambda, \mathbf{a}, \mathbf{b})$ *associated with stable matching market* $(I, J, \{\geq_i\}_{i\in I}, \{\geq_j\}_{j\in J})$*. Then* μ *is a Pareto-stable matching in the stable matching market.*

Proof. Since the outcome $(\mu, \mathbf{u}, \mathbf{v})$ is stable in the associated generalized assignment game, Lemma 3.3 implies that the matching μ is weakly stable in the stable matching market. It remains to show that the matching μ is not Pareto-dominated.

Let μ' be a matching of the stable matching market such that $\mu' \geq_{I \cup J} \mu$. Then $\mu'(i) \geq_i \mu(i)$ and $\mu'(j) \geq_j \mu(j)$ for every man $i \in I$ and woman $j \in J$. Hence $a_{i,\mu'(i)} \geq a_{i,\mu(i)}$ and $b_{\mu'(j),j} \geq b_{\mu(j),j}$ for every man $i \in I$ and woman $j \in J$. Since $a_{i,\mu'(i)} \geq a_{i,\mu(i)}$ for every man $i \in I$, we have

$$\sum_{i \in I} \Big(u_i \exp_\lambda(-a_{i,\mu'(i)}) - \pi_i \Big) \leq \sum_{i \in I} \Big(u_i \exp_\lambda(-a_{i,\mu(i)}) - \pi_i \Big).$$

Applying Lemma 3.4 to both sides, we get

$$\sum_{j \in J} \Big(b_{\mu'(j),j} N - v_j \Big) \leq \sum_{j \in J} \Big(b_{\mu(j),j} N - v_j \Big).$$

Since $b_{\mu'(j),j} \geq b_{\mu(j),j}$ for every woman $j \in J$, the inequalities are tight. Hence $a_{i,\mu'(i)} = a_{i,\mu(i)}$ and $b_{\mu'(j),j} = b_{\mu(j),j}$ for every man $i \in I$ and woman $j \in J$. Thus $\mu(i) \geq_i \mu'(i)$ and $\mu(j) \geq_j \mu'(j)$ for every man $i \in I$ and woman $j \in J$. We conclude that $\mu \geq_{I \cup J} \mu'$. $\square$

3.6 Group Strategyproofness

In this section, we study the group strategyproofness of matchings in the stable matching market that correspond to man-optimal outcomes in the associated generalized assignment game. First, in Lemma 3.6, we show that the utilities of the men in man-optimal outcomes in the associated generalized assignment game reflect the utilities of the men in the stable matching market. Then, in Theorem 3.7, we prove group strategyproofness in the stable matching market using group strategyproofness in the associated generalized assignment game.

Lemma 3.6. *Let* $(\mu, \mathbf{u}, \mathbf{v})$ *be a man-optimal outcome in the generalized assignment game* $(I, J, \boldsymbol{\pi}, N, \lambda, \mathbf{a}, \mathbf{b})$ *associated with stable matching market* $(I, J, \{\geq_i\}_{i \in I}, \{\geq_j\}_{j \in J})$*. Then* $\exp_\lambda(a_{i,\mu(i)}) \leq u_i < \exp_\lambda(a_{i,\mu(i)} + 1)$ *for every man* $i \in I$*.*

Proof. Since the outcome $(\mu, \mathbf{u}, \mathbf{v})$ is individually rational in the associated generalized assignment game, Lemma 3.1 implies that $u_i < \exp_\lambda(a_{i,\mu(i)}+1)$ for every man $i \in I$. It remains only to prove the lower bound $\exp_\lambda(a_{i,\mu(i)}) \leq u_i$ for every man $i \in I$.

Let $I' = \{i \in I \colon u_i < \exp_\lambda(a_{i,\mu(i)})\}$. For the sake of contradiction, suppose I' is nonempty. Let $J' = \{j \in J \colon j = \mu(i)$ for some man $i \in I'\}$. Notice that for every man $i \in I'$, we have $\mu(i) \neq 0$, for otherwise $u_i = \pi_i \exp_\lambda(a_{i,0}) \geq \exp_\lambda(a_{i,0})$ by the feasibility of $(\mu, \mathbf{u}, \mathbf{v})$. Thus J' is nonempty. Let

$$J'' = \{j \in J \colon 0 < \beta N + \pi_i - v_j < 1 \text{ for some man } i \in I' \text{ and } \beta \in \mathbb{Z} \text{ such that } \beta \geq 0\}.$$

Notice that for every man $i \in I'$ and woman $j \in J'$ such that $j = \mu(i)$, we have

$$0 < b_{i,j} N + \pi_i - v_j < 1$$

because Lemma 3.1 and the definition of I' imply

$$0 < u_i \exp_\lambda(-a_{i,j}) < 1,$$

and the feasibility and stability of $(\mu, \mathbf{u}, \mathbf{v})$ imply

$$u_i \exp_\lambda(-a_{i,j}) + v_j - (b_{i,j} N + \pi_i) = 0.$$

Thus $J' \subseteq J''$. Also, for every woman $j \in J''$, we have $v_j \neq b_{0,j} N$ by a simple non-integrality argument.

So, by applying Lemma 2.1 on J'', there exists a man $i' \in I$ and a woman $j'' \in J''$ such that $\mu(i') \notin J''$ and

$$u_{i'} \exp_\lambda(-a_{i',j''}) + v_{j''} - (b_{i',j''}N + \pi_{i'}) = 0. \tag{3.1}$$

Since $j'' \in J''$, there exists $i'' \in I'$ and $\beta'' \in \mathbb{Z}$ such that $\beta'' \geq 0$ and

$$0 < \beta'' N + \pi_{i''} - v_{j''} < 1. \tag{3.2}$$

Let $j' = \mu(i')$. We have

$$u_{i'} = \begin{cases} (b_{i',j'}N + \pi_{i'} - v_{j'}) \exp_\lambda(a_{i',j'}) & \text{if } j' \neq 0 \\ \pi_{i'} \exp_\lambda(a_{i',0}) & \text{if } j' = 0 \end{cases} \tag{3.3}$$

since $(\mu, \mathbf{u}, \mathbf{v})$ is stable. We consider two cases.

Case 1: $j' = 0$. Combining (3.1), (3.2), and (3.3), we get

$$0 < (\beta'' - b_{i',j''})N + (\pi_{i''} - \pi_{i'}) + \pi_{i'} \exp_\lambda(a_{i',0} - a_{i',j''}) < 1. \tag{3.4}$$

By a simple non-integrality argument, we have $a_{i',0} < a_{i',j''}$. Let $\Delta = \pi_{i'} \exp_\lambda(a_{i',0} - a_{i',j''})$. Since $a_{i',0} - a_{i',j''} \leq -1$, we have $0 < \Delta < 1$. Since (3.4) implies

$$0 < (\beta'' - b_{i',j''})N + (\pi_{i''} - \pi_{i'}) + \Delta < 1,$$

we have $\beta'' = b_{i',j''}$ and $\pi_{i''} = \pi_{i'}$. Thus $i'' = i'$ by the distinctness of $\boldsymbol{\pi}$. Since $i' = i'' \in I'$, we have $\mu(i') \in J' \subseteq J''$, which is a contradiction.

Case 2: $\mu(i') \neq 0$. Combining (3.1), (3.2), and (3.3), we get

$$0 < (\beta'' - b_{i',j''})N + (\pi_{i''} - \pi_{i'}) + (b_{i',j'}N + \pi_{i'} - v_{j'}) \exp_\lambda(a_{i',j'} - a_{i',j''}) < 1. \tag{3.5}$$

We consider three subcases.

Case 2.1: $a_{i',j'} \leq a_{i',j''} + 1$. Let $\Delta = (b_{i',j'}N + \pi_{i'} - v_{j'}) \exp_\lambda(a_{i',j'} - a_{i',j''})$. Then

$$\Delta \leq (b_{i',j'}N + \pi_{i'} - v_{j'})\lambda^{-1} \leq (b_{i',j'}N + \pi_{i'} - 0)\lambda^{-1} < 1,$$

where the second inequality follows from Lemma 3.1. Also, (3.3) implies

$$\Delta = u_{i'} \exp_\lambda(-a_{i',j''}) > 0,$$

where the inequality follows from Lemma 3.1. Hence $0 < \Delta < 1$. Since (3.5) implies

$$0 < (\beta'' - b_{i',j''})N + (\pi_{i''} - \pi_{i'}) + \Delta < 1,$$

we have $\beta'' = b_{i',j''}$ and $\pi_{i''} = \pi_{i'}$. So, $i'' = i'$ by the distinctness of $\boldsymbol{\pi}$. Since $i' = i'' \in I'$, we have $\mu(i') \in J' \subseteq J''$, which is a contradiction.

Case 2.2: $a_{i',j'} = a_{i',j''}$. Substituting into (3.5), we get

$$0 < (\beta'' - b_{i',j''} + b_{i',j'})N + \pi_{i''} - v_{j'} < 1. \tag{3.6}$$

This shows that

$$\beta'' - b_{i',j''} + b_{i',j'} \leq -1, \tag{3.7}$$

for otherwise $\mu(i') = j' \in J''$. Combining (3.7) with the lower bound in (3.6), we get

$$0 \leq -N + \pi_{i''} - v_{j'} < -v_{j'},$$

which contradicts Lemma 3.1.

Case 2.3: $a_{i',j'} \geq a_{i',j''} + 1$. The upper bound in (3.5) gives

$$\begin{aligned} b_{i',j'}N + \pi_{i'} - v_{j'} &< ((b_{i',j''} - \beta'')N + \pi_{i'} - \pi_{i''} + 1)\exp_\lambda(a_{i',j''} - a_{i',j'}) \\ &\leq ((b_{i',j''} - 0)N + \pi_{i'} - 1 + 1)\exp_\lambda(-1) \\ &= (b_{i',j''}N + \pi_{i'})\lambda^{-1} \\ &< 1. \end{aligned}$$

This shows that

$$b_{i',j'}N + \pi_{i'} - v_{j'} < 0, \tag{3.8}$$

for otherwise $\mu(i') = j' \in J''$. Combining (3.8) with (3.3), we get

$$u_{i'} \exp_\lambda(-a_{i',j'}) < 0,$$

which contradicts Lemma 3.1. $\square$

Since the compensation received by a man $i \in I$ matched with a woman $\mu(i) \neq 0$ is given by $u_i \exp_\lambda(-a_{i,\mu(i)})$, Lemma 3.6 implies that the amount of compensation in man-optimal outcomes is at least 1 and at most λ. In fact, no woman is willing to pay more than λ under any individual rational outcome.

Theorem 3.7 (Group strategyproofness)**.** *If a mechanism produces matchings that correspond to man-optimal outcomes of the generalized assignment game associated with the stable matching markets, then it is Pareto-stable and group strategyproof for the men.*

Proof. We have shown Pareto-stability in the stable matching market in Theorem 3.5. It remains only to show group strategyproofness.

Let $(I, J, \{\geq_i\}_{i\in I}, \{\geq_j\}_{j\in J})$ and $(I, J, \{\geq'_i\}_{i\in I}, \{\geq_j\}_{j\in J})$ be stable matching markets where $\{\geq_i\}_{i\in I}$ and $\{\geq'_i\}_{i\in I}$ are different preference profiles. Let $(I, J, \boldsymbol{\pi}, N, \lambda, \mathbf{a}, \mathbf{b})$ and $(I, J, \boldsymbol{\pi}, N, \lambda, \mathbf{a}', \mathbf{b})$ be the generalized assignment games associated with the stable matching markets $(I, J, \{\geq_i\}_{i\in I}, \{\geq_j\}_{j\in J})$ and $(I, J, \{\geq'_i\}_{i\in I}, \{\geq_j\}_{j\in J})$, respectively. Let $(\mu, \mathbf{u}, \mathbf{v})$ and $(\mu', \mathbf{u}', \mathbf{v}')$ be man-optimal outcomes of the associated generalized assignment games $(I, J, \boldsymbol{\pi}, N, \lambda, \mathbf{a}, \mathbf{b})$ and $(I, J, \boldsymbol{\pi}, N, \lambda, \mathbf{a}', \mathbf{b})$, respectively.

Consider $I' = \{i \in I\colon \text{the preference relations } \geq_i \text{ and } \geq'_i \text{ are different}\}$. By Theorem 2.2, there exists a man $i' \in I'$ such that $\mu'(i') \neq 0$ implies $u_{i'} \exp_\lambda(-a_{i',\mu'(i')}) \geq u'_{i'} \exp_\lambda(-a'_{i',\mu'(i')})$. It suffices to show that $\mu(i') \geq_{i'} \mu'(i')$. We consider two cases.

Case 1: $\mu'(i') = 0$. Then individual rationality of μ implies $\mu(i') \geq_{i'} 0 = \mu'(i')$.

Case 2: $\mu'(i') \neq 0$. Then

$$\begin{aligned}
\exp_\lambda(a_{i',\mu(i')} + 1) &> u_{i'} \\
&\geq \frac{u'_{i'}}{\exp_\lambda(a'_{i',\mu'(i')})} \exp_\lambda(a_{i',\mu'(i')}) \\
&\geq \exp_\lambda(a_{i',\mu'(i')}),
\end{aligned}$$

where the first and third inequalities follow from Lemma 3.6. This shows that $a_{i',\mu(i')} + 1 > a_{i',\mu'(i')}$. Hence $a_{i',\mu(i')} \geq a_{i',\mu'(i')}$, and we conclude that $\mu(i') \geq_{i'} \mu'(i')$. □

The implementation of our group strategyproof Pareto-stable mechanism amounts to computing a man-optimal outcome for the associated generalized assignment game. Since all utility functions in the associated generalized assignment game are linear functions, we can perform this computation using the algorithm of Dütting et al. [24], which was developed for multi-item auctions. If we model each woman j as a non-dummy item in the multi-item auction with price given by utility v_j, then the utility function of each man on each non-dummy item is a linear function of the price with a negative slope. Using the algorithm of Dütting et al., we can compute a man-optimal outcome using $O(n^5)$ arithmetic operations, where n is the total number of agents. Since poly(n) precision is sufficient, our mechanism admits a polynomial-time implementation.

For the purpose of solving the stable matching problem, it is actually sufficient for a mechanism to produce the matching without the utility vectors $\mathbf{u}$ and $\mathbf{v}$ of the associated generalized assignment game. This can be computed in $O(n^4)$ time using the equivalence of our mechanism and a process based on iterated unit-demand auctions [18].

Chapter 4

Maximum Stable Matchings

In this chapter, we study the problem of finding large weakly stable matchings. Our main result is a polynomial-time algorithm that achieves an approximation ratio of $1 + (1 - \frac{1}{L})^L$ for maximum stable matching with one-sided ties and incomplete lists where the lengths of the ties are at most L. Our algorithm is based on a proposal process in which numerical priorities are adjusted according to the solution of a linear program. This result extends the $1 + \frac{1}{e}$ approximation ratio for one-sided ties with unbounded lengths established in our conference paper [63] using a variant of this algorithm.

In Section 4.1, we review the prior work related to this problem. In Section 4.2, we present an overview of our techniques. In Section 4.3, we provide some formal definitions. In Section 4.4, we present the linear programming formulation used by our algorithm. In Section 4.5, we present the proposal process and the implementations of our algorithm. In Section 4.6, we analyze the approximation ratio of our algorithm.

4.1 Related Work

In Section 4.1.1, we review the prior work on the maximum stable matching problem where ties are allowed on both sides of the market. In Section 4.1.2, we review the prior work for the case where ties are only allowed on one side. In Section 4.1.3, we review the prior work for the case where the maximum tie length is restricted. In Section 4.1.4, we mention other special cases of maximum stable matching that have been studied in the literature. In Section 4.1.5, we mention recent work on strategic issues associated with approximation algorithms for the maximum stable matching problem.

4.1.1 Two-Sided Ties

The problem of finding large weakly stable matchings with ties and incomplete lists has been intensively studied. When either ties or incomplete lists are absent, all weakly stable matchings have the same size [34, 82]. However, when both ties and incomplete lists are present, weakly stable matchings can vary in size.

It is straightforward to see that any weakly stable matching is a 2-approximate solution [68]. Using a local search approach, Iwama et al. [51] gave an algorithm with an approximation ratio of $\frac{15}{8}$ ($= 1.875$). Király [58] improved the approximation ratio to $\frac{5}{3}$ (≈ 1.6667) by introducing the idea of promoting unmatched agents to higher priorities for tie-breaking. The current best approximation ratio for two-sided ties and incomplete lists is $\frac{3}{2}$ ($= 1.5$), which is attained by the polynomial-time algorithm of McDermid [70], and the linear-time algorithms of Paluch [75] and of Király [59].

For hardness results, Iwama et al. [50] were the first to prove that finding a maximum weakly stable matching with ties and incomplete lists is NP-hard. Halldórsson et al. [37] showed that it is NP-hard to get an approximation ratio of $1+\varepsilon$. Results by Yanagisawa [97] imply that getting an approximation ratio of $\frac{33}{29} - \varepsilon$ (≈ 1.1379) is NP-hard, and that of $\frac{4}{3} - \varepsilon$ (≈ 1.3333) is UG-hard. These hardness results hold even when the maximium tie length is two.

In the case of two-sided ties, Iwama et al. [52] showed that the integrality gap for the associated linear programming formulation is at least $\frac{3L-2}{2L-1}$, where L is the maximum tie length. For the case of unbounded tie lengths, this implies a lower bound of $\frac{3}{2}$ for the integrality gap, which coincides with the best approximation ratio known [59, 70, 75], indicating a potential barrier to further improvements.

4.1.2 One-Sided Ties

For the case where ties appear only on one side of the market, Király [58] showed an approximation ratio of $\frac{3}{2}$ ($= 1.5$) for an algorithm based on the idea of promotion, and conjectured that a $(\frac{3}{2} - \varepsilon)$-approximation is UG-hard. However, Iwama et al. [52] later presented an algorithm based on linear programming with an approximation ratio of $\frac{25}{17}$ (≈ 1.4706). Dean and Jalasutram [15] improved on this approach to obtain an approximation ratio of $\frac{19}{13}$ (≈ 1.4615). Huang and Kavitha [46] established an approximation ratio of $\frac{22}{15}$ (≈ 1.4667) using an algorithm based on rounding half-integral stable matchings. Subsequently, a tight analysis [6, 80] of their algorithm established an approximation ratio of $\frac{13}{9}$ (≈ 1.4444).

With one-sided ties, the problem of finding a maximum weakly stable matching remains NP-hard [68]. Results by Halldórsson et al. [39] imply that getting an approximation ratio of $\frac{21}{19} - \varepsilon$ (≈ 1.1053) is NP-hard, and that achieving $\frac{5}{4} - \varepsilon$ (≈ 1.25) is UG-hard. These hardness results hold even when each tie has length at most two.

In the case of one-sided ties, Iwama et al. [52] showed that the integrality gap for the associated linear programming formulation is at least $1 + (1 - \frac{1}{L})^L$, where L is the maximum tie length. For the case of unbounded tie lengths, this implies a lower bound of $1 + \frac{1}{e}$ (≈ 1.3679) for the integrality gap. In a paper by Huang et al. [45], the integrality gap is claimed to be at least $\frac{3}{2}$, but as pointed out in our conference paper [63], their proof contains an error.

4.1.3 Ties with Restricted Lengths

For the case of two-sided ties where the length of each tie is at most two, Halldórsson et al. [39] presented an algorithm based on checking a small subset of tie breakers that achieves an approximation ratio of $\frac{13}{7}$ (≈ 1.8571). For the same special case, the randomized algorithm of Halldórsson et al. [38] gives an expected approximation ratio of $\frac{7}{4}$ ($= 1.75$). Huang and Kavitha [46] established an approximation ratio of $\frac{10}{7}$ (≈ 1.4286) using the approach of rounding half-integral stable matchings. A better analysis [11] of their algorithm improved the approximation ratio to $\frac{4}{3}$ (≈ 1.3333), which matches the UG-hardness result [97] and the lower bound of the integral gap [52].

For the case of one-sided ties, the deterministic algorithm of Halldórsson et al. [39] attains an approximation ratio of $\frac{2}{1+L^{-2}}$, where L is the maximum length of the ties. The randomized algorithm of Halldórsson et al. [38] attains an approximation ratio of $\frac{10}{7}$ (≈ 1.4286) for the case of one-sided ties where the length of each tie is at most two.

4.1.4 Other Special Cases

Further known NP-hard problems include the case where ties are restricted to the tail of the preference lists [39], where preference lists have length at most three [48], where preferences are symmetric [4], or where preferences are derived from master lists [49]. Some parameterized complexity results are also known [69].

4.1.5 Strategyproofness

Recently, Hamada et al. [40] studied the strategic issues related to approximation algorithms for the maximum stable matching problem. For the case where ties appear only on one side, they showed that no $(\frac{3}{2} - \varepsilon)$-approximation algorithm is strategyproof for the side with ties, and no $(2 - \varepsilon)$-approximation algorithm is strategyproof for the side without ties. They also showed that these bounds are tight even for group strategyness mechanisms.

4.2 Overview of the Techniques

The key techniques for the design and analysis of our approximation algorithm are based on linear programming. We focus on the maximum stable matching problem with one-sided ties and incomplete lists, and obtain a polynomial-time $(1 + (1 - \frac{1}{L})^L)$-approximation algorithm, where L is the maximum tie length.

Our algorithm is motivated by a proposal process similar to that of Iwama et al. [52], and that of Dean and Jalasutram [15], in which numerical priorities are adjusted according to the linear programming solution, and are used for tie-breaking purposes. However, instead of using their priority manipulation schemes, we introduce a method of priority incrementation based on an adjustable step size parameter. We first present the description and the properties of our process in terms of the step size parameter. We then consider the limit of this process as the step size becomes infinitesimally small, and present a polynomial-time algorithm which satisfies the key properties with the step size parameter set to zero.

Using these key properties, we analyze the approximation ratio of our algorithm by directly comparing the size of our output matching with the optimal value of the linear program. Although this is a standard approach to analyze approximation algorithms, it has not been used in prior work on this problem. Prior analyses [6, 46, 58, 80] which are not based on linear programming consider the symmetric difference of the output matching and an unknown optimal matching, and count augmenting paths of various lengths. Such symmetric difference arguments are also used in the analyses of Iwama et al. [52], and Dean and Jalasutram [15], where the output matching is compared to both an unknown optimal matching and an optimal linear programming solution. Instead of focusing on the symmetric difference, we develop a scheme that assigns charges to the matched man-woman pairs based on an exchange function. By applying the stability constraint and the tie-breaking criterion to the charges incurred due to indifferences in the preferences, we show that the charges cover the value of the linear programming solution. While none of the prior analyses directly implies an upper bound for the integrality gap, our approach enables us to obtain an upper bound of $1 + (1 - \frac{1}{L})^L$ for the integrality gap, where L is the maximum tie length. This matches the known lower bound for the integral gap [52]. When the maximum length of the ties is two, our result implies an approximation ratio and integrality gap of $\frac{5}{4}$ $(= 1.25)$, which matches the known UG-hardness result [39].

As part of our analysis, we formulate an infinite-dimensional factor-revealing linear program to find a good exchange function. The finite-dimensional factor-revealing linear programming technique was introduced by Jain et al. [53], and since then a number of variants have been proposed [29, 66, 72]. However, it is often difficult to obtain a nice closed-form solution. For the maximum stable matching problem with one-sided ties and incomplete lists, Dean and Jalasutram [15] obtained an approximation ratio of $\frac{19}{13}$ by enumerating the combinatorial structures of augmenting paths and resorting to a computer-assisted proof for solving a large factor-revealing linear program. In contrast, our infinite-dimensional factor-revealing linear program is derived from our charging argument. Using numerical results as guidance, we are able to obtain an an analytical solution for our infinite-dimensional factor-revealing linear program.

The analysis presented in this dissertation simplifies and extends the proofs in our conference paper [63]. Our conference paper considers only the case of one-sided ties without any restriction on the length of the ties, and establishes an approximation ratio and integrality gap of $1 + \frac{1}{e}$. For one-sided ties with a small maximum length L, we show in this dissertation an improved approximation ratio and integrality gap of $1 + (1 - \frac{1}{L})^L$. In the limit as L tends to infinity, this implies the same ratio of $1 + \frac{1}{e}$. The charging argument in the conference paper is also replaced with a new one that is more easy to understand.

4.3 Preliminary Definitions

Recall that a tie in the preference list of an agent $k \in I \cup J$ is an equivalence class of size at least 2 with respect to the equivalence relation $=_k$. We consider the case where ties only appear on one side of the stable matching market. Without loss of generality, we assume that ties only appear on the preference lists of the women in the rest of this chapter. Moreover, we also assume that there is at least one tie in the stable matching market, for otherwise every stable matching has the same size.

For every woman $j \in J$, we define the length of a tie in the preference list of woman j as the size of the equivalence class corresponding to the tie. We use L to denote the maximum length of the ties in the preference lists of the women, where $2 \leq L \leq |I| + 1$.

The goal of the maximum stable matching problem is to find a maximum-cardinality weakly stable matching for a given stable matching market. We say that a polynomial-time algorithm is a k-approximation algorithm, or has an approximation ratio of k, where $k \geq 1$, if the algorithm produces a weakly stable matching with cardinality at least $\frac{1}{k}$ times that of the largest weakly stable matching. We also say that a linear program has an integrality gap of $k \geq 1$ if k is the minimum ratio such that the objective value of an optimal fractional solution is at most k times that of an optimal integral solution.

4.4 The Linear Programming Formulation

The following linear programming formulation is based on that of Rothblum [90], which extends that of Vande Vate [95].

$$
\begin{array}{llll}
\text{maximize} & \displaystyle\sum_{(i,j)\in I\times J} x_{i,j} & & \\
\text{subject to} & \displaystyle\sum_{j\in J} x_{i,j} \leq 1 & \forall i \in I & \text{(C1)} \\
& \displaystyle\sum_{i\in I} x_{i,j} \leq 1 & \forall j \in J & \text{(C2)} \\
& \displaystyle\sum_{\substack{j'\in J \\ j'>_i j}} x_{i,j'} + \sum_{\substack{i'\in I \\ i'\geq_j i}} x_{i',j} \geq 1 & \forall (i,j) \in I \times J \text{ such that } j >_i 0 \text{ and } i >_j 0 & \text{(C3)} \\
& x_{i,j} = 0 & \forall (i,j) \in I \times J \text{ such that } 0 >_i j \text{ or } 0 >_j i & \text{(C4)} \\
& x_{i,j} \geq 0 & \forall (i,j) \in I \times J & \text{(C5)}
\end{array}
$$

For the model with strict preferences and incomplete lists, it is known [90] that an integral solution $\mathbf{x} = \{x_{i,j}\}_{(i,j)\in I\times J}$ corresponds to the indicator variables of a weakly stable matching if and only if $\mathbf{x}$ satisfies constraints (C1)–(C5). Our model allows ties to appear on the preference lists of the women. We also allow a woman to be indifferent between being unmatched and being matched with some of the men. Accordingly, we provide a proof of Lemma 4.1 for the sake of completeness.

Lemma 4.1. *An integral solution* $\mathbf{x}$ *corresponds to the indicator variables of a weakly stable matching if and only if it satisfies constraints* (C1)–(C5)*.*

Proof. Suppose $\mathbf{x}$ satisfies constraints (C1)–(C5). Constraints (C1), (C2), and (C5) imply that $\mathbf{x}$ corresponds to a valid matching μ. Constraint (C4) implies that μ is individually rational. To show the weak stability of μ, consider man $i \in I$ and woman $j \in J$. It suffices to show that (i, j) is not a strongly blocking pair. We may assume that $j >_i 0$ and $i >_j 0$, for otherwise individual rationality implies $\mu(i) \geq_i 0 \geq_i j$ or $\mu(j) \geq_j 0 \geq_j i$. Consider constraint (C3) associated with (i, j). At least one of the two summations is equal to 1. If the first summation is equal to 1, then $\mu(i) >_i j$. If the second summation is equal to 1, then $\mu(j) \geq_j i$. Thus, μ is a weakly stable matching.

Conversely, suppose $\mathbf{x}$ corresponds to a weakly stable matching μ. Since μ is a valid matching, constraints (C1), (C2), and (C5) are satisfied. Also, the individual rationality of μ implies that constraint (C4) is satisfied. To show that constraint (C3) is satisfied, consider $(i, j) \in I \times J$ such that $j >_i 0$ and $i >_j 0$. It suffices to show that at least one of the two summations in constraint (C3) associated with (i, j) is equal to 1. By the weak stability of μ, we have either $\mu(i) \geq_i j$ or $\mu(j) \geq_j i$. We consider two cases.

Case 1: $\mu(j) \geq_j i$. Since $\mu(j) \geq_j i >_j 0$, the second summation is equal to 1.

Case 2: $i >_j \mu(j)$ and $\mu(i) \geq_i j$. Since $i >_j \mu(j)$, we have $(i, j) \notin \mu$. Since $\mu(i) \geq_i j$ and $(i, j) \notin \mu$, we have $\mu(i) >_i j$. Since $\mu(i) >_i j >_i 0$, the first summation is equal to 1. $\square$

Given $\mathbf{x}$ which satisfies constraints (C1)–(C5), it is useful to define auxiliary variables

$$y_{i,j} = \sum_{\substack{j' \in J \\ j \geq_i j'}} x_{i,j'}$$

for every $(i, j) \in I \times (J \cup \{0\})$, and

$$z_{i,j} = \sum_{\substack{i' \in I \\ i >_j i'}} x_{i',j}$$

for every $(i, j) \in (I \cup \{0\}) \times J$.

Lemma 4.2. *The auxiliary variables satisfy the following conditions.*

(1) For every $i \in I$, we have $y_{i,0} = 0$.

(2) For every $i \in I$ and $j \in J$, we have $x_{i,j} \leq y_{i,j} \leq 1$.

(3) For every $i \in I$ and $j, j' \in J$ such that $j >_i j'$, we have $y_{i,j} \geq x_{i,j} + y_{i,j'}$.

(4) For every $i, i' \in I \cup \{0\}$ and $j \in J$ such that $i =_j i'$, we have $z_{i,j} = z_{i',j}$.

(5) For every $i \in I$ and $j \in J$ such that $j \geq_i 0$ and $i \geq_j 0$, we have $y_{i,j} + z_{i,j} \leq 1$.

Proof.

(1) Let $i \in I$. Then the definition of $y_{i,0}$ implies

$$y_{i,0} = \sum_{\substack{j' \in J \\ 0 \geq_i j'}} x_{i,j'} = \sum_{\substack{j' \in J \\ 0 >_i j'}} x_{i,j'} = 0,$$

where the last equality follows from constraint (C4).

(2) Let $i \in I$ and $j \in J$. By the definition of $y_{i,j}$, we have

$$y_{i,j} = \sum_{\substack{j' \in J \\ j \geq_i j'}} x_{i,j'} \geq x_{i,j},$$

where the inequality follows from constraint (C5). Also by the definition of $y_{i,j}$, we have

$$y_{i,j} = \sum_{\substack{j' \in J \\ j \geq_i j'}} x_{i,j'} \leq \sum_{j' \in J} x_{i,j'} \leq 1,$$

where the first inequality follows from constraint (C5), and the second inequality follows from constraint (C1).

(3) Let $i \in I$ and $j, j' \in J$ such that $j >_i j'$. Then the definitions of $y_{i,j}$ and $y_{i,j'}$ imply

$$y_{i,j} = \sum_{\substack{j'' \in J \\ j \geq_i j''}} x_{i,j''} \geq x_{i,j} + \sum_{\substack{j'' \in J \\ j' \geq_i j''}} x_{i,j''} = x_{i,j'} + y_{i,j'}.$$

(4) Let $i, i' \in I \cup \{0\}$ and $j \in J$ such that $i =_j i'$. Then the definitions of $z_{i,j}$ and $z_{i,j'}$ imply

$$z_{i,j} = \sum_{\substack{i'' \in I \\ i >_j i''}} x_{i'',j} = \sum_{\substack{i'' \in I \\ i' >_j i''}} x_{i'',j} = z_{i',j},$$

where the second equality follows from $i =_j i'$.

(5) Let $(i, j) \in I \times J$ such that $j \geq_i 0$ and $i \geq_j 0$. We consider two cases.

Case 1: $i =_j 0$. Then the definition of $z_{i,j}$ implies

$$z_{i,j} = \sum_{\substack{i' \in I \\ i >_j i'}} x_{i',j} = \sum_{\substack{i' \in I \\ 0 >_j i'}} x_{i',j} = 0 \leq 1 - y_{i,j},$$

where the second equality follows from $i =_j 0$, the third equality follows from constraint (C4), and the last inequality follows from part (2).

Case 2: $i >_j 0$. Since $j \in J$ and $j \geq_i 0$, we have $j >_i 0$. Since $j >_i 0$ and $i >_j 0$, constraints (C1)–(C3) imply

$$\begin{aligned}
0 &\leq \Big(1 - \sum_{j \in J} x_{i,j}\Big) + \Big(1 - \sum_{i \in I} x_{i,j}\Big) + \Big(-1 + \sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'} + \sum_{\substack{i' \in I \\ i' \geq_j i}} x_{i',j}\Big) \\
&= 1 - \sum_{\substack{j' \in J \\ j \geq_i j'}} x_{i,j'} - \sum_{\substack{i' \in I \\ i >_j i'}} x_{i',j} \\
&= 1 - y_{i,j} - z_{i,j},
\end{aligned}$$

where the last equality follows from the definitions of $y_{i,j}$ and $z_{i,j}$. □

4.5 The Algorithm

In Section 4.5.1, we present a proposal process along with some key properties in terms of a step size parameter. In Section 4.5.2, we present a polynomial-time algorithm to simulate this process with an infinitesimally small step size. In Section 4.5.3, we present some properties of the loop body of our algorithm. In Section 4.5.4, we show that our algorithm satisfies the key properties. In Section 4.5.5, we present an alternative implementation of our algorithm.

4.5.1 A Proposal Process with Priorities

Our proposal process with priorities takes a stable matching market and a step size parameter $\eta > 0$ as input, and produces a weakly stable matching μ as output. In the preprocessing phase, we compute an optimal fractional solution $\mathbf{x}$ to the associated linear program. Then, in the initialization phase, we assign the empty matching to μ and each man i is assigned a priority p_i equal to 0. For each man i, we also maintain a set S_i of women that is initialized to the empty set. We use the set S_i to store the women to whom man i must propose before his priority p_i is increased by η. After that, the process enters the proposal phase and proceeds iteratively.

In each iteration, we pick an unmatched man i with priority $p_i < 1 + \eta$. If the set S_i is empty, we increment his priority p_i by η and then update S_i to the set

$$\{j \in J \colon j \geq_i 0 \text{ and } p_i \geq 1 - y_{i,j}\}.$$

Otherwise, the man i that we pick has a non-empty set S_i of women. Let j denote the most preferred woman of man i in S_i. We remove j from S_i and man i proposes to woman j. When woman j receives the proposal from man i, she tentatively accepts him if she is currently unmatched and he is acceptable to her. Otherwise, if woman j is currently matched to another man i', she tentatively accepts her preferred choice between men i and i', and rejects the other. In the event of a tie, she compares the current priorities p_i and $p_{i'}$ of the men and accepts the one with higher priority. (If the priorities of i and i' are equal, she breaks the tie arbitrarily.) If man i is tentatively accepted by woman j, the matching μ is updated accordingly.

When every unmatched man i has priority $p_i \geq 1 + \eta$, the process terminates and outputs the final matching μ.

Our process is similar to that of Iwama et al. [52], and that of Dean and Jalasutram [15], which also use a proposal scheme with priorities. In particular, the way that we populate the set S_i with a subset of women by referring to the linear programming solution is based on their methods. The major difference is that, in our process, priorities only increase by a small step size η, whereas in their algorithms, the priorities may increase by a possibly larger amount, essentially to ensure that a new woman is added to S_i. As in their algorithms, for every woman j, the sequence of tentative partners $\mu(j)$ of woman j satisfies a natural monotonicity property. Woman j is initially unmatched, and becomes matched the first time she receives a proposal from a man who is acceptable to her. In each subsequent iteration, she either keeps her current partner or gets a weakly preferred partner. Furthermore, if she is indifferent between her new partner and her old partner, then the new partner has a weakly larger priority. When the process terminates, the following properties hold, which are analogous to properties satisfied by the algorithms of Iwama et al. [52] and Dean and Jalasutram [15].

(P1) Let $(i, j) \in \mu$. Then $j \geq_i 0$ and $i \geq_j 0$.

(P2) Let $i \in I$ be a man and $j \in J$ be a woman such that $j \geq_i \mu(i)$ and $i \geq_j 0$. Then $\mu(j) \neq 0$ and $\mu(j) \geq_j i$.

(P3) Let $i \in I$ be a man. Then $1 - y_{i,\mu(i)} \leq p_i \leq 1 + 2\eta$.

(P4) Let $i \in I$ be a man and $j \in J$ be a woman such that $j \geq_i 0$ and $i \geq_j 0$. Suppose $p_i - \eta > 1 - y_{i,j}$. Then $\mu(j) \neq 0$ and $\mu(j) \geq_j i$. Furthermore, if $\mu(j) =_j i$, then $p_{\mu(j)} \geq p_i - \eta$.

For (P1), it is easy to see that man i proposes to woman j only if she is acceptable to him, and woman j accepts a proposal from man i only if he is acceptable to her. For (P2), if man i weakly prefers woman j to $\mu(i)$ and is acceptable to woman j, then man i has proposed to woman j. Thus the monotonicity property implies that $\mu(j) \neq 0$ and $\mu(j) \geq_j i$. For (P3), it is easy to see that the priority p_i of man i lies within the specified range when he proposes to woman $\mu(i)$. For (P4), if man i and woman j satisfy the stated assumptions, then man i proposed to woman j when his priority was equal to $p_i - \eta$, and this proposal was eventually rejected. Immediately after this proposal was rejected, woman j was matched with a man i' such that $i' \neq i$ and $i' \geq_j i$. The monotonicity property implies that $\mu(j) \neq 0$ and $\mu(j) \geq_j i' \geq_j i$. Furthermore, if $\mu(j) =_j i$, then $\mu(j) =_j i' =_j i$. Since $i' =_j i$, the priority of man i' was at least $p_i - \eta$ when the aforementioned proposal was rejected. Since $\mu(j) =_j i'$, the monotonicity property implies that $p_{\mu(j)} \geq p_i - \eta$.

4.5.2 A Polynomial-Time Implementation

The proposal process with priorities of Section 4.5.1 depends on a step size parameter $\eta > 0$. To obtain a good approximation ratio, we would like the step size parameter η to be small. However, the running time of a naive implementation grows in proportion to η^{-1}. We can imagine that if we take an infinitesimally small step size, then (P1)–(P4) can be satisfied with $\eta = 0$.

Our algorithm is motivated by the idea of simulating the process of Section 4.5.1 with an infinitesimally small step size. We maintain for every man $i \in I$ a priority p_i and a pointer $\ell_i \in J \cup \{0\}$ into the preference list of man i. For every man $i \in I$ and woman $j \in J$, we think of man i as having proposed to woman j if and only if $j >_i \ell_i$ and $j \geq_i 0$. Given $\boldsymbol{\ell} = \{\ell_i\}_{i \in I}$ and $j \in J$, we define

$$I_j(\boldsymbol{\ell}) = \{i \in I \colon j >_i \ell_i \text{ and } j \geq_i 0\}$$

as the set of all men i who have proposed to woman j. Given $\boldsymbol{\ell} = \{\ell_i\}_{i \in I}$, we define $G(\boldsymbol{\ell})$ as the bipartite graph with vertex set $I \cup J$ and edge set

$$E(\boldsymbol{\ell}) = \{(i, j) \in I \times J \colon i \in I_j(\boldsymbol{\ell}) \text{ and } i \geq_j i' \text{ for every } i' \in I_j(\boldsymbol{\ell}) \cup \{0\}\}.$$

Given $\boldsymbol{\ell}$ and μ, we say that a (possibly zero-length) path π in $G(\boldsymbol{\ell})$ is μ-alternating if it alternates between edges not in μ and edges in μ. We say that a μ-alternating path π is oriented from k to k' if no edge in $\pi \cap \mu$ is incident to vertex k. The details of the implementation are given in Algorithm 4.1.

Algorithm 4.1 Maximum stable matching implemented with alternating paths

compute an optimal fractional solution $\mathbf{x}$ to the associated linear program
for every $(i, j) \in I \times (J \cup \{0\})$, let $w(i, j) = 1 - y_{i,j}$, where $y_{i,j}$ is defined as in Section 4.4
initialize μ to the empty matching
for every man $i \in I$, initialize ℓ_i to the most preferred $j \in J \cup \{0\}$ with respect to $\geq_i$
for every man $i \in I$, initialize p_i to $w(i, \ell_i)$
while there exists a man $i \in I$ such that $\mu(i) = 0$ and $\ell_i >_i 0$ **do**
    let i_0 be such a man, and let j_0 denote the woman ℓ_{i_0}
    update ℓ_{i_0} to the most preferred $j \in \{j' \in J \colon j_0 >_{i_0} j'\} \cup \{0\}$ with respect to $\geq_{i_0}$
    if $\mu(j_0) = 0$ and $i_0 \geq_{j_0} 0$ **then**
        update μ to $\mu \cup \{(i_0, j_0)\}$
    else
        let $i_1 = \begin{cases} \mu(j_0) & \text{if } i_0 >_{j_0} \mu(j_0) \text{ or } (i_0 =_{j_0} \mu(j_0) \text{ and } p_{i_0} > p_{\mu(j_0)}) \\ i_0 & \text{otherwise} \end{cases}$
        let $\mu_0 = (\mu \cup \{(i_0, j_0)\}) \setminus \{(i_1, j_0)\}$
        let I_0 denote $\{i \in I \colon i \text{ is reachable from } i_1 \text{ via a } \mu_0\text{-alternating path in } G(\boldsymbol{\ell})\}$
        let i_2 be a man in I_0 such that $w(i_2, \ell_{i_2}) = \min_{i \in I_0} w(i, \ell_i)$
        let π_0 be a μ_0-alternating path from i_1 to i_2 in $G(\boldsymbol{\ell})$
        update p_i to $\max(p_i, w(i_2, \ell_{i_2}))$ for each man i in I_0
        update μ to the symmetric difference $\mu_0 \oplus \pi_0$
    end if
end while
return matching μ

It is straightforward to prove that throughout any execution of Algorithm 4.1, the program variable μ corresponds to a matching. Likewise, where it is defined, the program variable μ_0 corresponds to a matching. Accordingly, throughout our analysis, we assume that μ and μ_0 are matchings.

4.5.3 The Loop Body of the Algorithm

In the subsection, we analyze the loop body of Algorithm 4.1. It is convenient to define the following predicates.

$\mathcal{Q}_1(\boldsymbol{\ell})$: for every $i \in I$, we have $\ell_i \geq_i 0$.

$\mathcal{Q}_2(\boldsymbol{\ell}, \mu)$: μ is a matching of $G(\boldsymbol{\ell})$ such that for every $i \in I$ and $j \in J$, if $i \in I_j(\boldsymbol{\ell})$ and $i \geq_j 0$, then $\mu(j) \neq 0$.

$\mathcal{Q}_3(\boldsymbol{\ell}, \mathbf{p})$: for every $i \in I$, we have $p_i \leq w(i, \ell_i)$.

$\mathcal{Q}_4(\boldsymbol{\ell}, \mu, \mathbf{p})$: for every $i \in I$ such that $\mu(i) = 0$, we have $p_i = w(i, \ell_i)$.

$\mathcal{Q}_5(\boldsymbol{\ell}, \mathbf{p})$: for every $i \in I$ and $j \in J$ such that $j >_i \ell_i$, we have $w(i, j) \leq p_i$.

$\mathcal{Q}_6(\boldsymbol{\ell}, \mu, \mathbf{p})$: for every $i, i' \in I$ and $j \in J$ such that $(i, j) \in E(\boldsymbol{\ell})$ and $\mu(i') = j$, we have $p_i \leq p_{i'}$.

$\mathcal{Q}(\boldsymbol{\ell}, \mu, \mathbf{p})$: all of $\mathcal{Q}_1(\boldsymbol{\ell})$, $\mathcal{Q}_2(\boldsymbol{\ell}, \mu)$, $\mathcal{Q}_3(\boldsymbol{\ell}, \mathbf{p})$, $\mathcal{Q}_4(\boldsymbol{\ell}, \mu, \mathbf{p})$, $\mathcal{Q}_5(\boldsymbol{\ell}, \mathbf{p})$, and $\mathcal{Q}_6(\boldsymbol{\ell}, \mu, \mathbf{p})$ hold.

Consider the loop body of Algorithm 4.1. Throughout the rest of this subsection, we let $\boldsymbol{\ell}^-$, μ^-, and $\mathbf{p}^-$ denote the values of $\boldsymbol{\ell}$, μ, and $\mathbf{p}$ before the iteration such that $\mathcal{Q}(\boldsymbol{\ell}^-, \mu^-, \mathbf{p}^-)$ and the loop condition are satisfied. Also, we also let $\boldsymbol{\ell}^+$, μ^+, $\mathbf{p}^+$ denote the values of $\boldsymbol{\ell}$, μ, and $\mathbf{p}$ after the iteration.

Lemma 4.3. *For every $i \in I$, we have $\ell_i^+ \geq_i 0$.*

Proof. The only line in the loop body that modifies $\boldsymbol{\ell}$ is line 8, which updates ℓ_{i_0}. The definition of i_0 implies that $\ell_{i_0}^- >_{i_0} 0$. It follows that $\ell_{i_0}^+ \geq_{i_0} 0$ holds. $\square$

The following lemma characterizes how $E(\boldsymbol{\ell})$ changes in a single iteration of the loop of Algorithm 4.1. We omit the proof, which is straightforward but tedious.

Lemma 4.4. *The following conditions hold.*

(1) For every $j \in J$, we have $\mu^-(j) \geq_j 0$.

(2) If $i_0 <_{j_0} \mu^-(j_0)$, then $E(\boldsymbol{\ell}^+) = E(\boldsymbol{\ell}^-)$.

(3) If $\mu^-(j_0) = 0$ and $i_0 \geq_{j_0} 0$, then $E(\boldsymbol{\ell}^+) = E(\boldsymbol{\ell}^-) \cup \{(i_0, j_0)\}$.

(4) If $\mu^-(j_0) \neq 0$ and $i_0 =_{j_0} \mu^-(j_0)$, then $E(\boldsymbol{\ell}^+) = E(\boldsymbol{\ell}^-) \cup \{(i_0, j_0)\}$.

(5) If $\mu^-(j_0) \neq 0$ and $i_0 >_{j_0} \mu^-(j_0)$, then $E(\boldsymbol{\ell}^+) = \{(i, j) \in E(\boldsymbol{\ell}^-) \colon j \neq j_0\} \cup \{(i_0, j_0)\}$.

Lemma 4.5. *Condition $\mathcal{Q}_2(\boldsymbol{\ell}^+, \mu^+)$ holds. Furthermore, if $\mu^-(j_0) \neq 0$ or $0 >_{j_0} i_0$, then $\mathcal{Q}_2(\boldsymbol{\ell}^+, \mu_0)$ holds.*

Proof. Since $\mathcal{Q}_2(\boldsymbol{\ell}^-, \mu^-)$ holds, we know that μ^- is a matching of $G(\boldsymbol{\ell}^-)$. Let $i \in I$ and $j \in J$ be such that $i \in I_j(\boldsymbol{\ell}^+)$ and $i \geq_j 0$.

Case 1: $\mu^-(j_0) = 0$ and $i_0 \geq_{j_0} 0$. Then $\mu^+ = \mu^- \cup \{(i_0, j_0)\}$. Since $\mu^-(j_0) = 0$, $i_0 \geq_{j_0} 0$, and $\mathcal{Q}_2(\boldsymbol{\ell}^-, \mu^-)$ holds, part (3) of Lemma 4.4 implies that $E(\boldsymbol{\ell}^+) = E(\boldsymbol{\ell}^-) \cup \{(i_0, j_0)\}$. Since μ^- is a matching of $G(\boldsymbol{\ell}^-)$, $\mu^-(i_0) = 0$, $\mu^-(j_0) = 0$, $E(\boldsymbol{\ell}^+) = E(\boldsymbol{\ell}^-) \cup \{(i_0, j_0\}$, and

$\mu^+ = \mu^- \cup \{(i_0, j_0)\}$, we find that μ^+ is a matching of $G(\boldsymbol{\ell}^+)$. To establish that $\mathcal{Q}_2(\boldsymbol{\ell}^+, \mu_0)$ holds, it remains to prove that $\mu^+(j) \neq 0$.

Case 1.1: $j \neq j_0$. Then $I_j(\boldsymbol{\ell}^+) = I_j(\boldsymbol{\ell}^-)$, and hence $i \in I_j(\boldsymbol{\ell}^-)$. Since $i \in I_j(\boldsymbol{\ell}^-)$, $i \geq_j 0$, and $\mathcal{Q}_2(\boldsymbol{\ell}^-, \mu^-)$ holds, we have $\mu^-(j) \neq 0$. Since $\mu^+ = \mu^- \cup \{(i_0, j_0)\}$ and $j \neq j_0$, we have $\mu^+(j) = \mu^-(j)$. Since $\mu^+(j) = \mu^-(j)$ and $\mu^-(j) \neq 0$, we have $\mu^+(j) \neq 0$.

Case 1.2: $j = j_0$. Since $\mu^+ = \mu^- \cup \{(i_0, j_0)\}$, μ^+ is a matching of $G(\boldsymbol{\ell}^+)$, and $j = j_0$, we deduce that $\mu^+(j) = i_0 \neq 0$.

Case 2: $\mu^-(j_0) \neq 0$ or $0 >_{j_0} i_0$. We need to prove that $\mathcal{Q}_2(\boldsymbol{\ell}^+, \mu_0)$ and $\mathcal{Q}_2(\boldsymbol{\ell}^+, \mu^+)$ hold. We begin by establishing two useful claims.

The first claim is that μ_0 is a matching of $G(\boldsymbol{\ell}^+)$ that matches the same set of women as μ^-. To prove this claim, we consider three cases.

(a) $i_0 <_{j_0} \mu^-(j_0)$. Then $i_1 = i_0$ and part (2) of Lemma 4.4 implies $E(\boldsymbol{\ell}^+) = E(\boldsymbol{\ell}^-)$. Since $i_1 = i_0$, we have $\mu_0 = \mu^-$. Since $\mu_0 = \mu^-$, $E(\boldsymbol{\ell}^+) = E(\boldsymbol{\ell}^-)$, and μ^- is a matching of $G(\boldsymbol{\ell}^-)$, the claim follows.

(b) $i_0 =_{j_0} \mu^-(j_0)$. Then part (4) of Lemma 4.4 implies $E(\boldsymbol{\ell}^+) = E(\boldsymbol{\ell}^-) \cup \{(i_0, j_0)\}$. Since $\mu_0 = (\mu^- \cup \{(i_0, j_0)\}) \setminus \{(i_1, j_0)\}$, $E(\boldsymbol{\ell}^+) = E(\boldsymbol{\ell}^-) \cup \{(i_0, j_0)\}$, and μ^- is a matching of $G(\boldsymbol{\ell}^-)$, the claim follows.

(c) $i_0 >_{j_0} \mu^-(j_0)$. Then $i_1 \neq i_0$ and part (5) of Lemma 4.4 implies $E(\boldsymbol{\ell}^+) = \{(i, j) \in E(\boldsymbol{\ell}^-) \colon j \neq j_0\} \cup \{(i_0, j_0)\}$. Since $\mu_0 = (\mu^- \cup \{(i_0, j_0)\}) \setminus \{(i_1, j_0)\}$, $E(\boldsymbol{\ell}^+) = \{(i, j) \in E(\boldsymbol{\ell}^-) \colon j \neq j_0\} \cup \{(i_0, j_0)\}$, and μ^- is a matching of $G(\boldsymbol{\ell}^-)$, the claim follows.

The second claim is that μ^+ is a matching of $G(\boldsymbol{\ell}^+)$ that matches the same set of women as μ_0. Since μ_0 is a matching of $G(\boldsymbol{\ell}^+)$ and μ^+ is the symmetric difference between μ_0 and an oriented μ_0-alternating path in $G(\boldsymbol{\ell}^+)$ from i_1 to i_2, the second claim follows.

Given the two preceding claims, we can establish that $\mathcal{Q}_2(\boldsymbol{\ell}^+, \mu_0)$ and $\mathcal{Q}_2(\boldsymbol{\ell}^+, \mu^+)$ hold by proving that $\mu^-(j) \neq 0$. If $j = j_0$, the latter inequality follows from the Case 2 condition. Now suppose that $j \neq j_0$. Then $I_j(\boldsymbol{\ell}^+) = I_j(\boldsymbol{\ell}^-)$, and hence $i \in I_j(\boldsymbol{\ell}^-)$. Since $i \in I_j(\boldsymbol{\ell}^-)$, $i \geq_j 0$, and $\mathcal{Q}_2(\boldsymbol{\ell}^-, \mu^-)$ holds, we have $\mu^-(j) \neq 0$. $\square$

Lemma 4.6. *For every $i \in I$, we have $p_i^+ \leq w(i, \ell_i^+)$.*

Proof. Let $i \in I$. We consider two cases.

Case 1: $p_i^+ = p_i^-$. Since $\mathcal{Q}_3(\boldsymbol{\ell}^-, \mathbf{p}^-)$ holds, we have $p_i^- \leq w(i, \ell_i^-)$. Line 8 of Algorithm 4.1 implies $w(i, \ell_i^-) \leq w(i, \ell_i^+)$. Thus $p_i^+ = p_i^- \leq w(i, \ell_i^-) \leq w(i, \ell_i^+)$.

Case 2: $p_i^+ \neq p_i^-$. Then line 17 of Algorithm 4.1 implies $i \in I_0$ and $p_i^+ = w(i_2, \ell_{i_2}^+)$. Since $i \in I_0$, line 15 of Algorithm 4.1 implies $w(i_2, \ell_{i_2}^+) \leq w(i, \ell_i^+)$. Thus $p_i^+ = w(i_2, \ell_{i_2}^+) \leq w(i, \ell_i^+)$. $\square$

Lemma 4.7. *For every $i \in I$ such that $\mu^+(i) = 0$, we have $p_i^+ = w(i, \ell_i^+)$.*

Proof. Let $i \in I$ be such that $\mu^+(i) = 0$. Then Lemma 4.6 implies that $p_i^+ \leq w(i, \ell_i^+)$. It remains to show that $p_i^+ \geq w(i, \ell_i^+)$. We consider two cases.

Case 1: $\mu^-(j_0) = 0$ and $i_0 \geq_{j_0} 0$. Then $p_i^+ = p_i^-$. Since $\mu^+(i) = 0$, line 10 of Algorithm 4.1 implies $i \neq i_0$ and $\mu^-(i) = 0$. Since $\mu^-(i) = 0$, condition $\mathcal{Q}_4(\boldsymbol{\ell}^-, \mu^-, \mathbf{p}^-)$ implies $p_i^- = w(i, \ell_i^-)$. Since $i \neq i_0$, line 8 of Algorithm 4.1 implies $\ell_i^+ = \ell_i^-$. Thus $p_i^+ = p_i^- = w(i, \ell_i^-) = w(i, \ell_i^+)$.

Case 2: $\mu^-(j_0) \neq 0$ or $0 >_{j_0} i_0$. We consider two subcases.

Case 2.1: $i = i_2$. Then line 15 of Algorithm 4.1 implies $i_2 \in I_0$. Since $i = i_2 \in I_0$, line 17 of Algorithm 4.1 implies $p_i^+ \geq w(i, \ell_i^+)$.

Case 2.2: $i \neq i_2$. Since $\mu^+(i) = 0$, $i \neq i_2$, and $\{i' \in I \colon \mu^+(i') \neq 0\} = (\{i' \in I \colon \mu^-(i') \neq 0\} \cup \{i_0\}) \setminus \{i_2\}$, we deduce that $i \neq i_0$ and $\mu^-(i) = 0$. Line 17 of Algorithm 4.1 implies $p_i^+ \geq p_i^-$. Since $\mu^-(i) = 0$, condition $\mathcal{Q}_4(\boldsymbol{\ell}^-, \mu^-, \mathbf{p}^-)$ implies $p_i^- = w(i, \ell_i^-)$. Since $i \neq i_0$, line 8 of Algorithm 4.1 implies $\ell_i^+ = \ell_i^-$. Thus $p_i^+ \geq p_i^- = w(i, \ell_i^-) = w(i, \ell_i^+)$. $\square$

Lemma 4.8. *For every $i \in I$ and $j \in J$ such that $j >_i \ell_i^+$, we have $w(i,j) \leq p_i^+$.*

Proof. Let $i \in I$ and $j \in J$ be such that $j >_i \ell_i^+$. Line 17 of Algorithm 4.1 implies $p_i^+ \geq p_i^-$. We consider two cases.

Case 1: $j >_i \ell_i^-$. Then $\mathcal{Q}_5(\boldsymbol{\ell}^-, \mathbf{p}^-)$ implies $p_i^- \geq w(i,j)$. Thus $p_i^+ \geq p_i^- \geq w(i,j)$.

Case 2: $\ell_i^- \geq_i j$. Since $\ell_i^- \geq_i j >_i \ell_i^+$, line 8 of Algorithm 4.1 implies $i = i_0$ and $j = \ell_i^-$. Since $i = i_0$, line 7 of Algorithm 4.1 implies $\mu^-(i) = 0$. Since $\mu^-(i) = 0$, condition $\mathcal{Q}_4(\boldsymbol{\ell}^-, \mu^-, \mathbf{p}^-)$ implies $p_i^- = w(i, \ell_i^-)$. Thus $p_i^+ \geq p_i^- = w(i, \ell_i^-) = w(i,j)$. $\square$

Lemma 4.9. *Suppose that $\mu^-(j_0) \neq 0$ or $0 >_{j_0} i_0$. Then the following conditions hold.*

(1) For every $i, i' \in I$ and $j \in J$ such that $(i,j) \in E(\boldsymbol{\ell}^+)$ and $\mu_0(i') = j$, we have $p_i^- \leq p_{i'}^-$.

(2) For every $i, i' \in I$ and $j \in J$ such that $(i,j) \in E(\boldsymbol{\ell}^+)$ and $\mu_0(i') = j$, we have $p_i^+ \leq p_{i'}^+$.

(3) For every $i \in I$ on path π_0, we have $p_i^+ = w(i_2, \ell_{i_2}^+)$.

(4) For every $i, i' \in I$ and $j \in J$ such that $(i,j) \in E(\boldsymbol{\ell}^+)$ and $\mu^+(i') = j$, we have $p_i^+ \leq p_{i'}^+$.

Proof.

(1) Let $i, i' \in I$ and $j \in J$ be such that $(i, j) \in E(\boldsymbol{\ell}^+)$ and $\mu_0(i') = j$. We consider two cases.

Case 1: $j \neq j_0$. Since $j \neq j_0$, we have $\mu^-(j) = \mu_0(j) = i'$. In addition, Lemma 4.4 implies that $(i, j) \in E(\boldsymbol{\ell}^-)$. Since $\mu^-(j) = i'$ and $(i, j) \in E(\boldsymbol{\ell}^-)$, condition $\mathcal{Q}_6(\boldsymbol{\ell}^-, \mu^-, \mathbf{p}^-)$ implies $p_i^- \leq p_{i'}^-$.

Case 2: $j = j_0$. Thus $i' = \mu_0(j_0)$. Let $i'' \in I$ denote $\mu^-(j_0)$. We consider two subcases.

Case 2.1: $i \neq i_0$. Since $i \neq i_0$, Lemma 4.4 implies that $(i, j_0) \in E(\boldsymbol{\ell}^-)$. Since $(i, j_0) \in E(\boldsymbol{\ell}^-)$, condition $\mathcal{Q}_6(\boldsymbol{\ell}^-, \mu^-, \mathbf{p}^-)$ implies $p_i^- \leq p_{i''}^-$. Since $i \neq i_0$ and $(i, j_0) \in E(\boldsymbol{\ell}^+)$, Lemma 4.4 implies that $i \geq_{j_0} i_0$. Since $i \geq_{j_0} i_0$, lines 12 and 13 of Algorithm 4.1 imply that $p_{i''}^- \leq p_{i'}^-$. Thus $p_i^- \leq p_{i''}^- \leq p_{i'}^-$.

Case 2.2: $i = i_0$. Since $(i_0, j_0) \in E(\boldsymbol{\ell}^+)$, we have $i_0 \geq_{j_0} i''$. Since $i_0 \geq_{j_0} i''$, lines 12 and 13 of Algorithm 4.1 imply that $p_{i_0}^- \leq p_{i'}^-$.

(2) Let $i, i' \in I$ and $j \in J$ be such that $(i, j) \in E(\boldsymbol{\ell}^+)$ and $\mu_0(i') = j$. We consider two cases.

Case 1: $p_i^+ = p_i^-$. Then line 17 of Algorithm 4.1 implies $p_{i'}^+ \geq p_{i'}^-$. Part (1) implies $p_{i'}^- \geq p_i^-$. Thus $p_{i'}^+ \geq p_{i'}^- \geq p_i^- = p_i^+$.

Case 2: $p_i^+ \neq p_i^-$. Then line 17 of Algorithm 4.1 implies $i \in I_0$ and $p_i^+ = w(i_2, \ell_{i_2}^+)$. Since $i \in I_0$, line 14 of Algorithm 4.1 implies there exists an oriented μ_0-alternating path in $G(\boldsymbol{\ell}^+)$ from i_1 to i. Since $(i, j) \in E(\boldsymbol{\ell}^+)$ and $\mu_0(i') = j$, there exists an oriented μ_0-alternating path in $G(\boldsymbol{\ell}^+)$ from i to i'. Hence there exists an an oriented μ_0-alternating path in $G(\boldsymbol{\ell}^+)$ from i_1 to i'. So the line 14 of Algorithm 4.1 implies $i' \in I_0$. Since $i' \in I_0$, line 17 of Algorithm 4.1 implies $p^+_{i'} \geq w(i_2, \ell^+_{i_2})$. Thus $p^+_{i'} \geq w(i_2, \ell^+_{i_2}) = p^+_i$.

(3) Let $i_1 = i'_1, \ldots, i'_s = i_2$ denote the sequence of men on path π_0. By part (1), we have $p^-_{i'_t} \leq p^-_{i'_{t+1}}$ for $1 \leq t < s$. It follows that $p^-_i \leq p^-_{i_2}$ for every man i on path π_0. Since $\mathcal{Q}_3(\boldsymbol{\ell}^-, \mathbf{p}^-)$ holds, we have $p^-_{i_2} \leq w(i_2, \ell^-_{i_2}) \leq w(i_2, \ell^+_{i_2})$. Thus $p^-_i \leq w(i_2, \ell^+_{i_2})$ for every man i on path π_0. Since every man on path π_0 belongs to I_0, line 17 of Algorithm 4.1 implies that $p^+_i = w(i_2, \ell^+_{i_2})$ for every man i on path π_0.

(4) Let J' denote the set of women who are matched in μ_0. Line 18 of Algorithm 4.1 ensures that the set of women who are matched in μ^+ is also J'. Moreover, by part (3), $p^+_{\mu^+(j)} = p^+_{\mu_0(j)}$ for every woman j in J'. Consequently, part (2) implies that $\mathcal{Q}_6(\boldsymbol{\ell}^+, \mu^+, \mathbf{p}^+)$ holds. $\square$

Lemma 4.10. *Let $i, i' \in I$ and $j \in J$ be such that $(i,j) \in E(\boldsymbol{\ell}^+)$ and $\mu^+(i') = j$. Then $p^+_i \leq p^+_{i'}$.*

Proof. If $\mu^-(j_0) \neq 0$ or $0 >_{j_0} i_0$, then part (4) of Lemma 4.9 implies that $p^+_i \leq p^+_{i'}$. For the remainder of the proof, assume that $\mu^-(j_0) = 0$ and $i_0 \geq_{j_0} 0$. Thus $\mu^+ = \mu^- \cup \{(i_0, j_0)\}$, $\mathbf{p}^+ = \mathbf{p}^-$, and part (3) of Lemma 4.4 implies $E(\boldsymbol{\ell}^+) = E(\boldsymbol{\ell}^-) \cup \{(i_0, j_0)\}$. We consider two cases.

Case 1: $j \neq j_0$. Since $j \neq j_0$ and μ^+ is equal to $\mu^- \cup \{(i_0, j_0)\}$, we have $\mu^-(j) = \mu^+(j) = i'$. Since $(i,j) \in E(\boldsymbol{\ell}^+) = E(\boldsymbol{\ell}^-) \cup \{(i_0, j_0)\}$ and $j \neq j_0$, we have $(i,j) \in E(\boldsymbol{\ell}^-)$. Since $(i,j) \in E(\boldsymbol{\ell}^-)$, $\mu^-(i') = j$, and $\mathcal{Q}_6(\boldsymbol{\ell}^-, \mu^-, \mathbf{p}^-)$ holds, we have $p^-_i \leq p^-_{i'}$. Since $p^-_i \leq p^-_{i'}$ and $\mathbf{p}^+ = \mathbf{p}^-$, we have $p^+_i \leq p^+_{i'}$.

Case 2: $j = j_0$. Since $\mu^-(j_0) = 0$ and $\mathcal{Q}_2(\boldsymbol{\ell}^-, \mu^-)$ holds, we deduce that none of the edges in $E(\boldsymbol{\ell}^-)$ are incident on j_0. Since $j = j_0$ and none of the edges in $E(\boldsymbol{\ell}^-)$ are incident on j_0, we have $(i, j) \notin E(\boldsymbol{\ell}^-)$. Since $(i, j) \notin E(\boldsymbol{\ell}^-)$ and $(i, j) \in E(\boldsymbol{\ell}^+) = E(\boldsymbol{\ell}^-) \cup \{(i_0, j_0)\}$, we have $(i, j) = (i_0, j_0)$. Since $\mu^+ = \mu^- \cup \{(i_0, j_0)\}$, we have $\mu^+(j_0) = i_0$. Since $(i, j) = (i_0, j_0)$ and $\mu^+(j_0) = i_0$, we have $i' = \mu^+(j) = \mu^+(j_0) = i_0 = i$. Since $i = i'$ we have $p_i^+ = p_{i'}^+$. $\square$

Lemma 4.11. *Consider the loop body of Algorithm 4.1. Let $\boldsymbol{\ell}^-$, μ^-, and $\mathbf{p}^-$ denote the values of $\boldsymbol{\ell}$, μ, and $\mathbf{p}$ at the start of the iteration. Assume that the loop condition is satisfied, and that $\mathcal{Q}(\boldsymbol{\ell}^-, \mu^-, \mathbf{p}^-)$ holds. Let $\boldsymbol{\ell}^+$, μ^+, and $\mathbf{p}^+$ denote the values of $\boldsymbol{\ell}$, μ, and $\mathbf{p}$ at the end of the iteration. Then $\mathcal{Q}(\boldsymbol{\ell}^+, \mu^+, \mathbf{p}^+)$ holds.*

Proof. Lemma 4.3 implies that $\mathcal{Q}_1(\boldsymbol{\ell}^+)$ holds. Lemma 4.5 implies that $\mathcal{Q}_2(\boldsymbol{\ell}^+, \mu^+)$ holds. Lemma 4.6 implies that $\mathcal{Q}_3(\boldsymbol{\ell}^+, \mathbf{p}^+)$ holds. Lemma 4.7 implies that $\mathcal{Q}_4(\boldsymbol{\ell}^+, \mu^+, \mathbf{p}^+)$ holds. Lemma 4.8 implies that $\mathcal{Q}_5(\boldsymbol{\ell}^+, \mathbf{p}^+)$ holds. Lemma 4.10 implies that $\mathcal{Q}_6(\boldsymbol{\ell}^+, \mu^+, \mathbf{p}^+)$ holds. Thus $\mathcal{Q}(\boldsymbol{\ell}^+, \mu^+, \mathbf{p}^+)$ holds. $\square$

4.5.4 Correctness of the Algorithm

Lemma 4.12. *Let $\boldsymbol{\ell}$, μ, and $\mathbf{p}$ be such that $\mathcal{Q}(\boldsymbol{\ell}, \mu, \mathbf{p})$ holds. Suppose that for every $i \in I$, either $\mu(i) \neq 0$ or $0 \geq_i \ell_i$. Then $(\mu, \mathbf{p})$ satisfies (P1)–(P4) with $\eta = 0$.*

Proof. We begin by proving that (P1) holds. Let $(i, j) \in \mu$. Since $\mathcal{Q}_2(\boldsymbol{\ell}, \mu)$ holds, μ is matching of $G(\boldsymbol{\ell})$. Since $(i, j) \in \mu$ and μ is a matching of $G(\boldsymbol{\ell})$, we have $(i, j) \in E(\boldsymbol{\ell})$. Since $(i, j) \in E(\boldsymbol{\ell})$, we have $i \in I_j(\boldsymbol{\ell})$ and $i \geq_j 0$. Since $i \in I_j(\boldsymbol{\ell})$, we have $j >_i \ell_i$. Since $\mathcal{Q}_1(\boldsymbol{\ell})$ holds, we have $\ell_i \geq_i 0$. Since $j >_i \ell_i$ and $\ell_i \geq_i 0$, we have $j >_i 0$.

We now prove that (P2) holds. Let $i \in I$ be a man and $j \in J$ be a woman such that $j \geq_i \mu(i)$ and $i \geq_j 0$. We prove that $i \in I_j(\boldsymbol{\ell})$ by considering two cases.

Case 1: $\mu(i) = 0$. Then $0 \geq_i \ell_i$. Since $j \in J$ and $j \geq_i \mu(i) = 0$, we have $j >_i 0$. Since $j >_i 0 \geq_i \ell_i$, we have $i \in I_j(\boldsymbol{\ell})$.

Case 2: $\mu(i) \neq 0$. Since $(i, \mu(i)) \in \mu \subseteq E(\boldsymbol{\ell})$, we have $i \in I_j(\boldsymbol{\ell})$.

Having established that $i \in I_j(\boldsymbol{\ell})$, we now complete the proof that (P2) holds. Since $i \in I_j(\boldsymbol{\ell})$ and $i \geq_j 0$, condition $\mathcal{Q}_2(\boldsymbol{\ell}, \mu)$ implies $\mu(j) \neq 0$. Since $(\mu(j), j) \in E(\boldsymbol{\ell})$ and $i \in I_j(\boldsymbol{\ell})$, the definition of $E(\boldsymbol{\ell})$ implies $\mu(j) \geq_j i$.

We now prove that (P3) holds. Let $i \in I$ be a man. We consider two cases.

Case 1: $\mu(i) = 0$. Then $0 \geq_i \ell_i$. Since $\mathcal{Q}_1(\boldsymbol{\ell})$ holds, we have $\ell_i \geq_i 0$. Since $0 \geq_i \ell_i$ and $\ell_i \geq_i 0$, we have $\ell_i = 0$. Since $\ell_i = 0$, we have $w(i, \ell_i) = 1 - y_{i,\ell_i} = 1$. Since $\mu(i) = 0$, $w(i, \ell_i) = 1$, and $\mathcal{Q}_4(\boldsymbol{\ell}, \mu, \mathbf{p})$ holds, we have $p_i = 1$.

Case 2: $\mu(i) \neq 0$. Let j denote $\mu(i)$. Since $\mathcal{Q}_2(\boldsymbol{\ell}, \mu)$ holds, μ is a matching of $G(\boldsymbol{\ell})$. Since $\mu(i) = j$ and μ is a matching of $G(\boldsymbol{\ell})$, we have $(i, j) \in E(\boldsymbol{\ell})$. Since $(i, j) \in E(\boldsymbol{\ell})$, we have $i \in I_j(\boldsymbol{\ell})$ and hence $j >_i \ell_i$. Since $j >_i \ell_i$ and $\mathcal{Q}_5(\boldsymbol{\ell}, \mathbf{p})$ holds, we have $p_i \geq w(i, j) = 1 - y_{i,j}$. It remains to argue that $p_i \leq 1$. Since constraint (C1) holds, we have $w(i, \ell_i) \leq 1$. Since $w(i, \ell_i) \leq 1$ and $\mathcal{Q}_3(\boldsymbol{\ell}, \mathbf{p})$ holds, we have $p_i \leq 1$.

It remains to prove that (P4) holds. Let $i \in I$ be a man and $j \in J$ be a woman such that $j \geq_i 0$, $i \geq_j 0$, and $p_i > 1 - y_{i,j}$. Since $p_i > 1 - y_{i,j} = w(i, j)$ and $\mathcal{Q}_3(\boldsymbol{\ell}, \mathbf{p})$ holds, we have $j >_i \ell_i$ and hence $i \in I_j(\boldsymbol{\ell})$. Since $i \in I_j(\boldsymbol{\ell})$, $i \geq_j 0$, and $\mathcal{Q}_2(\boldsymbol{\ell}, \mu)$ holds, we know that μ is a matching of $G(\boldsymbol{\ell})$ with $\mu(j) \neq 0$. Let $i' \in I$ denote $\mu(j)$. Since μ is a matching of $G(\boldsymbol{\ell})$ and (i', j) belongs to μ, we have $(i', j) \in E(\boldsymbol{\ell})$. Since $(i', j) \in E(\boldsymbol{\ell})$ and $i \in I_j(\boldsymbol{\ell})$, the definition of $E(\boldsymbol{\ell})$ implies that $i' \geq_j i$. It remains to prove that if $i' =_j i$ then $p_i \leq p_{i'}$. Assume $i' =_j i$. Since $(i', j) \in E(\boldsymbol{\ell})$, $i \in I_j(\boldsymbol{\ell})$, and $i' =_j i$, the definition of $E(\boldsymbol{\ell})$ implies that $(i, j) \in E(\boldsymbol{\ell})$. Since $(i, j) \in E(\boldsymbol{\ell})$, $\mu(i') = j$, and $\mathcal{Q}_6(\boldsymbol{\ell}, \mu, \mathbf{p})$ holds, we have $p_i \leq p_{i'}$. $\square$

Lemma 4.13. *When Algorithm 4.1 terminates, $(\mu, \mathbf{p})$ satisfies (P1)–(P4) with $\eta = 0$.*

Proof. It is straightforward to verify that $\mathcal{Q}(\boldsymbol{\ell}, \mu, \mathbf{p})$ holds before the first iteration of the algorithm. So, by Lemma 4.11 and induction on the number of iterations, $\mathcal{Q}(\boldsymbol{\ell}, \mu, \mathbf{p})$ holds when the algorithm terminates. Moreover, line 6 implies that for every $i \in I$, we have $\mu(i) \neq 0$ or $0 \geq_i \ell_i$ when the algorithm terminates. Hence Lemma 4.12 implies that $(\mu, \mathbf{p})$ satisfies (P1)–(P4) with $\eta = 0$ when the algorithm terminates. $\square$

Lemma 4.14. *Let μ be a matching such that $(\mu, \mathbf{p})$ satisfies (P1) and (P2) for some $\mathbf{p}$. Then μ is a weakly stable matching.*

Proof. Since (P1) holds, μ is individually rational. To establish weak stability of μ, consider $(i, j) \in I \times J$. It suffices to show that (i, j) is not a strongly blocking pair. For the sake of contradiction, suppose $j >_i \mu(i)$ and $i >_j \mu(j)$. If $0 >_j i$, then $0 >_j i >_j \mu(j)$, contradicting the individual rationality of μ. If $i \geq_j 0$, then since $j >_i \mu(i)$, $i \geq_j 0$, and (P2) holds, we deduce that $\mu(j) \geq_j i$, contradicting the assumption that $i >_j \mu(j)$. $\square$

4.5.5 An Alternative Implementation

In this subsection, we present a more succinct alternative algorithm that does not maintain a priority vector $\mathbf{p}$. This alternative algorithm is implemented with weighted matchings.

Let us define the weight of any edge $(i, j) \in E(\boldsymbol{\ell})$ as $w(i, \ell_i)$. Also we define the weight of $\mu \subseteq E(\boldsymbol{\ell})$ as the total weight of all the edges in μ. We use the abbreviations MCM and MWMCM to denote the terms maximum-cardinality matching and maximum-weight MCM, respectively. Our algorithm iteratively updates $\boldsymbol{\ell}$ and computes an MWMCM in $G(\boldsymbol{\ell})$. The details of the implementation are given in Algorithm 4.2.

Algorithm 4.2 Maximum stable matching implemented with weighted matchings

```
compute an optimal fractional solution x to the associated linear program
for every (i, j) ∈ I × (J ∪ {0}), let w(i, j) = 1 − y_{i,j}, where y_{i,j} is defined as in Section 4.4
initialize μ to the empty matching
for every man i ∈ I, initialize ℓ_i to the most preferred j ∈ J ∪ {0} with respect to ≥_i
while there exists a man i ∈ I such that μ(i) = 0 and ℓ_i >_i 0 do
    let i_0 be such a man, and let j_0 denote the woman ℓ_{i_0}
    update ℓ_{i_0} to the most preferred j ∈ {j' ∈ J : j_0 >_{i_0} j'} ∪ {0} with respect to ≥_{i_0}
    update μ to an arbitrary MWMCM of G(ℓ)
end while
return matching μ
```

Lemma 4.15. *Let $\boldsymbol{\ell}$, μ, and $\mathbf{p}$ satisfy $\mathcal{Q}_6(\boldsymbol{\ell}, \mu, \mathbf{p})$, let $i, i' \in I$, and let π be an oriented μ-alternating path in $G(\boldsymbol{\ell})$ from i to i'. Then $p_i \leq p_{i'}$.*

Proof. If $i = i'$ then $p_i = p_{i'}$, so we can assume that $i \neq i'$. Let $i = i_1, i_2, \ldots, i_k = i'$ denote the sequence of $k > 1$ men appearing on path π. Since $\mathcal{Q}_6(\boldsymbol{\ell}, \mu, \mathbf{p})$ holds and π is an oriented μ-alternating path in $G(\boldsymbol{\ell})$ from i to i', we deduce that $p_{i_j} \leq p_{i_{j+1}}$ for all j such that $1 \leq j < k$. Hence $p_i = p_{i_1} \leq p_{i_k} = p_{i'}$. $\square$

Lemma 4.16. *Let $\boldsymbol{\ell}$, μ, and $\mathbf{p}$ satisfy $\mathcal{Q}_2(\boldsymbol{\ell}, \mu)$, $\mathcal{Q}_3(\boldsymbol{\ell}, \mathbf{p})$, $\mathcal{Q}_4(\boldsymbol{\ell}, \mu, \mathbf{p})$, and $\mathcal{Q}_6(\boldsymbol{\ell}, \mu, \mathbf{p})$. Then μ is an MWMCM of $G(\boldsymbol{\ell})$.*

Proof. Since $\mathcal{Q}_2(\boldsymbol{\ell}, \mu)$ holds, μ is an MCM of $G(\boldsymbol{\ell})$. Let μ' be an MWMCM of $G(\boldsymbol{\ell})$. Since μ' is an MCM of $G(\boldsymbol{\ell})$, $\mathcal{Q}_2(\boldsymbol{\ell}, \mu)$ implies that μ and μ' match the same set of women. Thus $\mu \oplus \mu'$ corresponds to a collection $\mathcal{X}$ of cycles (of positive even length) and man-to-man paths (of positive even length). For any cycle γ in $\mathcal{X}$, the edges of μ on γ match the same set of men as the edges of μ' on γ. Thus the total weight (in $G(\boldsymbol{\ell})$) of the edges of μ on γ is equal to the total weight of the edges of μ' on γ.

Now consider a man-to-man path π in $\mathcal{X}$. Let the endpoints of π be i and i', where i is matched in μ and not in μ', and i' is matched in μ' and not in μ. Since μ' is an MWMCM of $G(\boldsymbol{\ell})$, and since $\mu' \oplus \pi$ is an MCM of $G(\boldsymbol{\ell})$, we deduce that $w(i, \ell_i) \leq w(i', \ell_{i'})$. Since $\mu(i') = 0$ and $\mathcal{Q}_4(\boldsymbol{\ell}, \mu, \mathbf{p})$ holds, we have $p_{i'} = w(i', \ell_{i'})$. Since $\mathcal{Q}_6(\boldsymbol{\ell}, \mu, \mathbf{p})$ holds and π is an oriented μ-alternating path in $G(\boldsymbol{\ell})$ from i to i', Lemma 4.15 implies that $p_i \geq p_{i'}$. Since $\mathcal{Q}_3(\boldsymbol{\ell}, \mathbf{p})$ holds, we have $p_i \leq w(i, \ell_i)$. Since $p_i \geq p_{i'}$ and $p_i \leq w(i, \ell_i) \leq w(i', \ell_{i'}) = p_{i'}$, we deduce that $p_i = w(i, \ell_i) = w(i', \ell_{i'}) = p_{i'}$. Thus the total weight (in $G(\boldsymbol{\ell})$) of the edges of μ on π is equal to the total weight of the edges of μ' on π.

The foregoing analysis of the cycles and paths in $\mathcal{X}$ implies that the weight of μ is equal to that of μ', and hence that μ is an MWMCM of $G(\boldsymbol{\ell})$. $\square$

Lemma 4.17. *An invariant of the Algorithm 4.1 loop is that μ is an MWMCM of $G(\boldsymbol{\ell})$.*

Proof. It is easy to check that μ is an MWMCM of $G(\boldsymbol{\ell})$ when the Algorithm 4.1 loop is first encountered. Hence the claim of the lemma follows by Lemmas 4.11 and 4.16. $\square$

Lemma 4.18. *Let $\boldsymbol{\ell}$, μ, and $\mathbf{p}$ satisfy $\mathcal{Q}_2(\boldsymbol{\ell}, \mu)$, $\mathcal{Q}_3(\boldsymbol{\ell}, \mathbf{p})$, $\mathcal{Q}_4(\boldsymbol{\ell}, \mu, \mathbf{p})$, and $\mathcal{Q}_6(\boldsymbol{\ell}, \mu, \mathbf{p})$, and let μ' be an MWMCM of $G(\boldsymbol{\ell})$. Then $\mathcal{Q}_2(\boldsymbol{\ell}, \mu')$, $\mathcal{Q}_4(\boldsymbol{\ell}, \mu', \mathbf{p})$, and $\mathcal{Q}_6(\boldsymbol{\ell}, \mu', \mathbf{p})$ hold.*

Proof. Lemma 4.16 implies that μ is an MWMCM of $G(\boldsymbol{\ell})$. Let J' denote the set of women with nonzero degree in $G(\boldsymbol{\ell})$. Since $\mathcal{Q}_2(\boldsymbol{\ell}, \mu)$ holds, the set of women matched by μ is J'. Since μ' is an MCM, we deduce that the set of women matched by μ' is also J', and hence that $\mathcal{Q}_2(\boldsymbol{\ell}, \mu')$ holds. Thus $\mu \oplus \mu'$ corresponds to a collection $\mathcal{X}$ of cycles (of positive even length) and man-to-man paths (of positive even length).

Consider a cycle γ in $\mathcal{X}$. Since $\mathcal{Q}_6(\boldsymbol{\ell}, \mu, \mathbf{p})$ holds and there is an oriented μ-alternating path in $G(\boldsymbol{\ell})$ from i to i' for every pair of men i and i' on γ, Lemma 4.15 implies that $p_i = p_{i'}$ for all men i and i' on γ.

Consider a path π in $\mathcal{X}$. Let the endpoints of π be i and i', where i is matched in μ and not in μ', and i' is matched in μ' and not in μ. Since $\mathcal{Q}_6(\boldsymbol{\ell}, \mu, \mathbf{p})$ holds and π is an oriented μ-alternating path in $G(\boldsymbol{\ell})$ from i' to i, there are oriented μ-alternating paths in $G(\boldsymbol{\ell})$ from i' to i'' and from i'' to i for every man i'' on π. Thus Lemma 4.15 implies that $p_{i'} \leq p_{i''} \leq p_i$ for every man i'' on π. Since μ and μ' are each MWMCMs, and $\mu \oplus \pi$ and $\mu' \oplus \pi$ are MCMs of $G(\boldsymbol{\ell})$, we deduce that $w(i, \ell_i) = w(i', \ell_{i'})$. Since $\mathcal{Q}_4(\boldsymbol{\ell}, \mu, \mathbf{p})$ holds, we have $p_{i'} = w(i', \ell_{i'})$. Since $\mathcal{Q}_3(\boldsymbol{\ell}, \mathbf{p})$ holds, we have $p_i \leq w(i, \ell_i)$. Since $p_i \leq w(i, \ell_i) = w(i', \ell_{i'}) = p_{i'} \leq p_i$, we deduce that $p_i = w(i, \ell_i) = p_{i'}$. Since $p_i = w(i, \ell_i)$, we conclude that $\mathcal{Q}_4(\boldsymbol{\ell}, \mu', \mathbf{p})$ holds. Since $p_i = p_{i'}$ and $p_{i'} \leq p_{i''} \leq p_i$ for every man i'' on π, we deduce that $p_i = p_{i''}$ for every man i'' on π.

The foregoing analysis of the cycles and paths in $\mathcal{X}$ implies that $p_{\mu(j)} = p_{\mu'(j)}$ for every woman j in J'. Since $\mathcal{Q}_6(\boldsymbol{\ell}, \mu, \mathbf{p})$ holds, we deduce that $\mathcal{Q}_6(\boldsymbol{\ell}, \mu', \mathbf{p})$ holds. □

We now use our results concerning Algorithm 4.1 to reason about Algorithm 4.2. To do this, it is convenient to introduce a hybrid algorithm, which we define by modifying

Algorithm 4.1 as follows: At the end of each iteration of the while loop, update the matching μ to to an arbitrary MWMCM of $G(\boldsymbol{\ell})$.

Lemma 4.19. *Consider the loop body of the hybrid algorithm. Let $\boldsymbol{\ell}^-$, μ^-, and $\mathbf{p}^-$ denote the values of $\boldsymbol{\ell}$, μ, and $\mathbf{p}$ at the start of the iteration. Assume that the loop condition is satisfied, and that $\mathcal{Q}(\boldsymbol{\ell}^-, \mu^-, \mathbf{p}^-)$ holds. Let $\boldsymbol{\ell}^+$, μ^+, and $\mathbf{p}^+$ denote the values of $\boldsymbol{\ell}$, μ, and $\mathbf{p}$ at the end of the iteration. Then $\mathcal{Q}(\boldsymbol{\ell}^+, \mu^+, \mathbf{p}^+)$ holds.*

Proof. Lemma 4.11 implies that $\mathcal{Q}(\boldsymbol{\ell}, \mu, \mathbf{p})$ holds just before μ is updated to an arbitrary MWMCM of $G(\boldsymbol{\ell})$. Lemma 4.16 implies that μ is an MWMCM of $G(\boldsymbol{\ell})$ at this point in the execution. Thus Lemma 4.18 implies that $\mathcal{Q}_2(\boldsymbol{\ell}^+, \mu^+)$, $\mathcal{Q}_4(\boldsymbol{\ell}^+, \mu^+, \mathbf{p}^+)$, and $\mathcal{Q}_6(\boldsymbol{\ell}^+, \mu^+, \mathbf{p}^+)$ hold. Since $\mathcal{Q}(\boldsymbol{\ell}, \mu, \mathbf{p})$ holds just before μ is updated to an arbitrary MWMCM of $G(\boldsymbol{\ell})$, we conclude that $\mathcal{Q}_1(\boldsymbol{\ell}^+)$, $\mathcal{Q}_3(\boldsymbol{\ell}^+, \mathbf{p}^+)$, and $\mathcal{Q}_5(\boldsymbol{\ell}^+, \mathbf{p}^+)$ hold. Hence $\mathcal{Q}(\boldsymbol{\ell}^+, \mu^+, \mathbf{p}^+)$ holds, as required. □

The converse of the following lemma also holds, but we only need the stated direction.

Lemma 4.20. *Fix an execution of Algorithm 4.2, and let T denote the number of times the body of the loop is executed. For $0 \leq t \leq T$, let $\boldsymbol{\ell}^{(t)}$ and $\mu^{(t)}$ denote the values of the corresponding program variables after t iterations of the loop. Then there is a T-iteration execution of the hybrid algorithm such that, for $0 \leq t \leq T$, the program variables $\boldsymbol{\ell}$ and μ are equal to $\boldsymbol{\ell}^{(t)}$ and $\mu^{(t)}$, respectively, after t iterations of the loop.*

Proof. Observe that Algorithm 4.2 and the hybrid algorithm are equivalent in terms of their initialization of $\boldsymbol{\ell}$ and μ, and also in terms of the set of possible updates to $\boldsymbol{\ell}$ and μ associated with any given iteration. (While the hybrid algorithm also maintains a priority vector $\mathbf{p}$, this priority vector has no influence on the overall update applied to $\boldsymbol{\ell}$ and μ in a given iteration.) Given this observation, the claim of the lemma is straightforward to prove by induction on t. $\square$

Lemma 4.21. *When Algorithm 4.2 terminates, there exists* $\mathbf{p}$ *such that* $(\mu, \mathbf{p})$ *satisfies (P1)–(P4) with* $\eta = 0$.

Proof. Fix an execution of Algorithm 4.2, and let $\boldsymbol{\ell}^*$ and μ^* denote the final values of $\boldsymbol{\ell}$ and μ. Lemma 4.20 implies that there exists an execution of the hybrid algorithm with the same final values of $\boldsymbol{\ell}$ and μ. Fix such an execution of the hybrid algorithm, and let $\mathbf{p}^*$ denote the final value of $\mathbf{p}$. It is straightforward to verify that $\mathcal{Q}(\boldsymbol{\ell}, \mu, \mathbf{p})$ holds the first time the loop is reached in this execution of the hybrid algorithm. Thus, Lemma 4.19 implies that $\mathcal{Q}(\boldsymbol{\ell}^*, \mu^*, \mathbf{p}^*)$ holds. Moreover, line 6 implies that for every $i \in I$, we have $\mu^*(i) \neq 0$ or $0 \geq_i \ell_i^*$. Hence Lemma 4.12 implies that $(\mu^*, \mathbf{p}^*)$ satisfies (P1)–(P4) with $\eta = 0$. $\square$

4.6 Analysis of the Approximation Ratio

In this section, we analyze the approximation ratio and the integrality gap. Our analysis applies to both Algorithm 4.1 and 4.2. Throughout this section, whenever we mention $\mathbf{x}$ and μ, we are referring to their values when the algorithm terminates. Given $\mathbf{x}$, we let the auxiliary variables $\{y_{i,j}\}_{(i,j)\in I\times(J\cup\{0\})}$ and $\{z_{i,j}\}_{(i,j)\in(I\cup\{0\})\times J}$ be defined as in Section 4.4. By Lemmas 4.13 and 4.21, there exists $\mathbf{p}$ such that $(\mu, \mathbf{p})$ satisfies (P1)–(P4) with $\eta = 0$. We fix such priority values $\mathbf{p}$ throughout this section.

In Section 4.6.1, we describe a charging scheme which covers the value of the linear programming solution. In Section 4.6.2, we bound the charge incurred by each matched man-woman pair. In Section 4.6.3, we show that the approximation ratio is at most $1 + (1 - \frac{1}{L})^L$.

4.6.1 The Charging Argument

Our charging argument is based on an exchange function $h\colon [0,1] \times [0,1] \to \mathbb{R}$ which satisfies the following properties.

(H1) For every $\xi_1, \xi_2 \in [0,1]$, we have $0 = h(0, \xi_2) \le h(\xi_1, \xi_2) \le 1$.

(H2) For every $\xi_1, \xi_2 \in [0,1]$ such that $\xi_1 > \xi_2$, we have $h(\xi_1, \xi_2) = 1$.

(H3) The function $h(\xi_1, \xi_2)$ is non-decreasing in ξ_1 and non-increasing in ξ_2.

(H4) For every $\xi_1, \xi_2 \in [0,1]$, we have

$$L \cdot \int_{\xi_2 \cdot (1-1/L)}^{\xi_2} \Big(1 - h(\xi_1, \xi)\Big)\, d\xi \le \max(\xi_2 - \xi_1, 0).$$

Given an exchange function h that satisfies (H1)–(H4), our charging argument is as follows. For every $(i,j) \in I \times J$, we assign to man i a charge of

$$\theta_{i,j} = \int_0^{x_{i,j}} h(1 - p_i, y_{i,j} - \xi)\, d\xi$$

and to woman j a charge of

$$\phi_{i,j} = \begin{cases} 0 & \text{if } \mu(j) = 0 \text{ or } i >_j \mu(j) \\ x_{i,j} & \text{if } \mu(j) \neq 0 \text{ and } \mu(j) >_j i \\ x_{i,j} - \displaystyle\int_0^{x_{i,j}} h(1 - p_{\mu(j)}, 1 - z_{\mu(j),j} - \xi)\, d\xi & \text{if } \mu(j) \neq 0 \text{ and } \mu(j) =_j i \end{cases}$$

The following lemma shows that the charges are non-negative and cover the value of an optimal solution to the linear program. We prove this using the stability constraint in the linear program and the tie-breaking criterion of our algorithm.

Lemma 4.22. *Let $i \in I$ and $j \in J$. Then $\theta_{i,j}$ and $\phi_{i,j}$ satisfy the following conditions.*

(1) $\theta_{i,j} \geq 0$ and $\phi_{i,j} \geq 0$.

(2) $x_{i,j} \leq \theta_{i,j} + \phi_{i,j}$.

Proof.

(1) The definition of $\theta_{i,j}$ implies

$$\theta_{i,j} = \int_0^{x_{i,j}} h(1 - p_i, y_{i,j} - \xi)\, d\xi \geq 0,$$

where the inequality follows from (H1). Also, the definition of $\phi_{i,j}$ implies

$$\begin{aligned}
\phi_{i,j} &\geq \min\Big(0, x_{i,j}, x_{i,j} - \int_0^{x_{i,j}} h(1 - p_{\mu(j)}, 1 - z_{\mu(j),j} - \xi)\, d\xi\Big) \\
&\geq \min\Big(0, x_{i,j}, x_{i,j} - \int_0^{x_{i,j}} 1\, d\xi\Big) \\
&= 0,
\end{aligned}$$

where the second inequality follows from (H1).

(2) We consider two cases.

Case 1: $y_{i,j} \leq 1 - p_i$. Then (H3) implies

$$\begin{aligned}
0 &\leq \int_0^{x_{i,j}} \Big(h(1 - p_i, y_{i,j} - \xi) - h(1 - p_i, 1 - p_i - \xi)\Big)\, d\xi \\
&= \int_0^{x_{i,j}} \Big(h(1 - p_i, y_{i,j} - \xi) - 1\Big)\, d\xi \\
&= \theta_{i,j} - x_{i,j} \\
&\leq \theta_{i,j} + \phi_{i,j} - x_{i,j},
\end{aligned}$$

where the first equality follows from (H2), the second equality follows from the definition of $\theta_{i,j}$, and the last inequality follows from part (1).

Case 2: $y_{i,j} > 1 - p_i$. We may assume that $x_{i,j} \neq 0$, for otherwise part (1) implies $\theta_{i,j} + \phi_{i,j} \geq 0 = x_{i,j}$. Since $x_{i,j} \neq 0$, constraint (C4) implies $j \geq_i 0$ and $i \geq_j 0$. So (P4) with $\eta = 0$ implies $\mu(j) \neq 0$ and $\mu(j) \geq_j i$. We consider two subcases.

Case 2.1: $\mu(j) >_j i$. Then the definition of $\phi_{i,j}$ implies

$$0 = \phi_{i,j} - x_{i,j} \leq \theta_{i,j} + \phi_{i,j} - x_{i,j},$$

where the inequality follows from part (1).

Case 2.2: $\mu(j) =_j i$. Then (P4) with $\eta = 0$ implies $p_i \leq p_{\mu(j)}$. Also, since $\mu(j) =_j i$, parts (4) and (5) of Lemma 4.2 imply $z_{\mu(j),j} = z_{i,j} \leq 1 - y_{i,j}$. Since $p_i \leq p_{\mu(j)}$ and $y_{i,j} \leq 1 - z_{\mu(j),j}$, (H3) implies

$$0 \leq \int_0^{x_{i,j}} \Big(h(1 - p_i, y_{i,j} - \xi) - h(1 - p_{\mu(j)}, 1 - z_{\mu(j),j} - \xi) \Big)\, d\xi = \theta_{i,j} + \phi_{i,j} - x_{i,j},$$

where the equality follows from the definitions of $\theta_{i,j}$ and $\phi_{i,j}$. □

4.6.2 Bounding the Charges

To bound the approximation ratio, Lemma 4.22 implies that it is sufficient to bound the charges. In Lemma 4.23, we derive an upper bound for the charges incurred by a man using the strict ordering in his preferences. In Lemma 4.24, we derive an upper bound for the charges incurred by a woman due to indifferences using the constraint on the tie length. In Lemma 4.25, we derive an upper bound for the total charges incurred by a matched couple by combining the results of Lemmas 4.23 and 4.24.

Lemma 4.23. *Let $i \in I$ be a man. Then*

$$\sum_{j \in J} \theta_{i,j} \leq \int_0^1 h(1 - p_i, \xi)\, d\xi.$$

Proof. Let $j_1, \ldots, j_{|J|} \in J$ such that $j_{|J|} >_i j_{|J|-1} >_i \cdots >_i j_1$. Then part (2) of Lemma 4.2 implies $y_{i,j_{|J|}} \leq 1$. Also parts (2) and (3) of Lemma 4.2 imply

$$y_{i,j_k} - x_{i,j_k} \geq \begin{cases} 0 & \text{if } k = 1 \\ y_{i,j_{k-1}} & \text{if } 1 < k \leq |J| \end{cases} \tag{4.1}$$

for every $1 \leq k \leq |J|$. Hence the definitions of $\{\theta_{i,j_k}\}_{1 \leq k \leq |J|}$ imply

$$\begin{aligned} \theta_{i,j_k} &= \int_0^{x_{i,j_k}} h(1 - p_i, y_{i,j_k} - \xi)\, d\xi \\ &= \int_{y_{i,j_k} - x_{i,j_k}}^{y_{i,j_k}} h(1 - p_i, \xi)\, d\xi \\ &\leq \begin{cases} \displaystyle\int_0^{y_{i,j_k}} h(1 - p_i, \xi)\, d\xi & \text{if } k = 1 \\ \displaystyle\int_{y_{i,j_{k-1}}}^{y_{i,j_k}} h(1 - p_i, \xi)\, d\xi & \text{if } 1 < k \leq |J| \end{cases} \end{aligned}$$

for every $1 \leq k \leq |J|$, where the inequality follows from (4.1) and (H1). Thus

$$\begin{aligned} \sum_{j \in J} \theta_{i,j} = \sum_{1 \leq k \leq |J|} \theta_{i,j_k} &\leq \int_0^{y_{i,j_1}} h(1 - p_i, \xi)\, d\xi + \sum_{1 < k \leq |J|} \int_{y_{i,j_{k-1}}}^{y_{i,j_k}} h(1 - p_i, \xi)\, d\xi \\ &= \int_0^{y_{i,j_{|J|}}} h(1 - p_i, \xi)\, d\xi \\ &\leq \int_0^1 h(1 - p_i, \xi)\, d\xi, \end{aligned}$$

where the last inequality follows from $y_{i,j_{|J|}} \leq 1$ and (H1). $\square$

Lemma 4.24. *Let $j \in J$ be a woman such that $\mu(j) \neq 0$. Then*

$$\sum_{\substack{i \in I \\ \mu(j) =_j i}} \phi_{i,j} \leq \max(p_{\mu(j)} - z_{\mu(j),j}, 0).$$

Proof. Let

$$H(\xi') = \int_{1-z_{\mu(j),j}-\xi'}^{1-z_{\mu(j),j}} \Big(1 - h(1-p_{\mu(j)}, \xi)\Big)\, d\xi$$

for every $\xi' \in [0,1]$. Then (H1) and (H3) imply that H is concave and non-decreasing. Also (H4) implies

$$L \cdot H\Big(\frac{1-z_{\mu(j),j}}{L}\Big) = L \cdot \int_{(1-z_{\mu(j),j})(1-1/L)}^{1-z_{\mu(j),j}} \Big(1 - h(1-p_{\mu(j)}, \xi)\Big)\, d\xi \le \max(p_{\mu(j)} - z_{\mu(j),j}, 0). \quad (4.2)$$

Let $I' = \{i \in I \colon \mu(j) =_j i\}$. Then $|I'| \le L$ since L is the maximum tie-length. Let $i_1, \ldots, i_{|I'|} \in I$ such that $I' = \{i_1, \ldots, i_{|I'|}\}$. Let

$$\xi_k = \begin{cases} x_{i_k,j} & \text{if } 1 \le k \le |I'| \\ 0 & \text{if } |I'| < k \le L \end{cases}$$

for every $1 \le k \le L$. Then the definition of $z_{\mu(j),j}$ implies

$$1 - z_{\mu(j),j} = 1 - \sum_{\substack{i \in I \\ \mu(j) >_j i}} x_{i,j} \ge \sum_{i \in I} x_{i,j} - \sum_{\substack{i \in I \\ \mu(j) >_j i}} x_{i,j} \ge \sum_{\substack{i \in I \\ \mu(j) =_j i}} x_{i,j} = \sum_{1 \le k \le |I'|} x_{i_k,j} = \sum_{1 \le k \le L} \xi_k,$$

where the first inequality follows from constraint (C2), the second equality follows from the definitions of I' and $\{i_k\}_{1 \le k \le |I'|}$, and the third equality follows from the definitions of $\{\xi_k\}_{1 \le k \le L}$. Hence the monotonicity and concavity of H imply

$$L \cdot H\Big(\frac{1-z_{\mu(j),j}}{L}\Big) \ge L \cdot H\Big(\frac{1}{L}\sum_{1 \le k \le L} \xi_k\Big) \ge \sum_{1 \le k \le L} H(\xi_k). \quad (4.3)$$

Thus the definitions of the definitions of $\{\phi_{i,j}\}_{i\in I}$ imply

$$\begin{aligned}
\sum_{\substack{i\in I\\ \mu(j)=_j i}} \phi_{i,j} &= \sum_{\substack{i\in I\\ \mu(j)=_j i}} \Big(x_{i,j} - \int_0^{x_{i,j}} h(1-p_{\mu(j)}, 1-z_{\mu(j),j}-\xi)\,d\xi\Big)\\
&= \sum_{\substack{i\in I\\ \mu(j)=_j i}} \int_{1-z_{\mu(j),j}-x_{i,j}}^{1-z_{\mu(j),j}} \Big(1-h(1-p_{\mu(j)},\xi)\Big)\,d\xi\\
&= \sum_{\substack{i\in I\\ \mu(j)=_j i}} H(x_{i,j})\\
&= \sum_{1\le k\le |I'|} H(x_{i_k,j})\\
&= \sum_{1\le k\le L} H(\xi_k)\\
&\le L\cdot H\Big(\frac{1-z_{\mu(j),j}}{L}\Big)\\
&\le \max(p_{\mu(j)} - z_{\mu(j),j}, 0),
\end{aligned}$$

where the third equality follows from the definition of H, the fourth equality follows from the definitions of I' and $\{i_k\}_{1\le k\le|I'|}$, the fifth equality follows from the definitions of $\{\xi_k\}_{1\le k\le L}$, the first inequality follows from (4.3), and the second inequality follows from (4.2). □

Lemma 4.25. *Let $i \in I$ and $j \in J \cup \{0\}$ such that $\mu(i) = j$. Then the following conditions hold.*

(1) If $j \neq 0$, then

$$\sum_{j'\in J}\theta_{i,j'} + \sum_{i'\in I}\phi_{i',j} \le 1 + \int_{1-p_i}^{1} h(1-p_i,\xi)\,d\xi.$$

(2) If $j = 0$, then $\theta_{i,j'} = 0$ for every $j' \in J$.

Proof.

(1) Suppose $j \neq 0$. Then (P1) implies $j \geq_i 0$ and $i \geq_j 0$. So part (5) of Lemma 4.2 implies

$$z_{i,j} \leq 1 - y_{i,j} \leq p_i,$$

where the second inequality follows from (P3) with $\eta = 0$. So the definitions of $\{\phi_{i',j}\}_{i' \in I}$ imply

$$\sum_{i' \in I} \phi_{i',j} = \sum_{\substack{i' \in I \\ \mu(j) =_j i'}} \phi_{i',j} + \sum_{\substack{i' \in I \\ \mu(j) >_j i'}} x_{i',j} \leq \max(p_i - z_{i,j}, 0) + z_{i,j} = p_i, \tag{4.4}$$

where the first inequality follows from Lemma 4.24 and the definition of $z_{i,j}$, and the last equality follows from $p_i \geq z_{i,j}$. Also, by Lemma 4.23, we have

$$\begin{aligned}
\sum_{j' \in J} \theta_{i,j'} &\leq \int_0^1 h(1-p_i, \xi)\, d\xi \\
&= \int_0^{1-p_i} h(1-p_i, \xi)\, d\xi + \int_{1-p_i}^1 h(1-p_i, \xi)\, d\xi \\
&= \int_0^{1-p_i} 1\, d\xi + \int_{1-p_i}^1 h(1-p_i, \xi)\, d\xi \\
&= 1 - p_i + \int_{1-p_i}^1 h(1-p_i, \xi)\, d\xi, \qquad (4.5)
\end{aligned}$$

where the second equality follows from (H2). Combining (4.4) and (4.5) gives the desired inequality.

(2) Suppose $j = 0$. Let $j' \in J$. Since $\mu(i) = j = 0$, (P3) with $\eta = 0$ implies

$$1 \geq p_i \geq 1 - y_{i,j} = 1 - y_{i,0} = 1,$$

where the last equality follows from part (1) of Lemma 4.2. Hence the definition of $\theta_{i,j'}$ implies

$$\theta_{i,j'} = \int_0^{x_{i,j'}} h(1 - p_i, y_{i,j'} - \xi)\, d\xi = \int_0^{x_{i,j'}} h(0, y_{i,j'} - \xi)\, d\xi = 0,$$

where the second equality follows from $p_i = 1$, and the third equality follows from (H1). $\square$

4.6.3 The Approximation Ratio

To obtain the approximation ratio, it remains to pick a good exchange function h satisfying (H1)–(H4) such that the right hand side of part (1) of Lemma 4.25 is small. Using a technique similar to that presented in our conference paper [63], we can formulate this as an infinite-dimensional factor-revealing linear program. More specifically, we can minimize

$$\sup_{\xi_1 \in [0,1]} \int_{\xi_1}^{1} h(\xi_1, \xi)\, d\xi$$

over the set of all functions h which satisfies (H1)–(H4). Notice that the objective value and the constraints induced by (H1)–(H4) are linear in h. However, the space of all feasible solutions is infinite-dimensional. One possible approach to the infinite-dimensional factor-revealing linear program is to obtaining a numerical solution via a suitable discretization. Using the numerical results as guidance, we obtain the candidate exchange function

$$h(\xi_1, \xi_2) = \max\left(\{0\} \cup \left\{(1 - \tfrac{1}{L})^k \colon k \in \{0, 1, 2, \dots\} \text{ and } \xi_1 > \xi_2 \cdot (1 - \tfrac{1}{L})^k\right\}\right). \tag{4.6}$$

The following lemma provides a formal analytical proof that it satisfies (H1)–(H4) and achieves an objective value of $(1 - \frac{1}{L})^L$.

Lemma 4.26. *Let h be the function defined by* (4.6). *Then the following conditions hold.*

(1) The function h satisfies (H1)–(H4).

(2) For every $\xi_1 \in [0,1]$, we have $\displaystyle\int_{\xi_1}^{1} h(\xi_1,\xi)\,d\xi \le \Big(1-\frac{1}{L}\Big)^L$.

Proof.

(1) It is straightforward to see that (H1)–(H3) hold by inspecting the definition of h. To show that (H4) holds, let $\xi_1, \xi_2 \in [0,1]$. We consider three cases.

Case 1: $\xi_2 \le \xi_1$. Then

$$L \cdot \int_{\xi_2\cdot(1-1/L)}^{\xi_2} \Big(1 - h(\xi_1,\xi)\Big)\,d\xi = L \cdot \int_{\xi_2\cdot(1-1/L)}^{\xi_2} (1-1)\,d\xi = 0 = \max(\xi_2-\xi_1,0).$$

Case 2: $\xi_2 > \xi_1 = 0$. Then

$$L \cdot \int_{\xi_2\cdot(1-1/L)}^{\xi_2} \Big(1 - h(\xi_1,\xi)\Big)\,d\xi = L \cdot \int_{\xi_2\cdot(1-1/L)}^{\xi_2} (1-0)\,d\xi = \xi_2 = \max(\xi_2-\xi_1,0).$$

Case 3: $\xi_2 > \xi_1 > 0$. Let $k \in \{0,1,2,\dots\}$ such that $(1-\frac{1}{L})^{k+1} < \frac{\xi_1}{\xi_2} \le (1-\frac{1}{L})^k$. Then

$$\begin{aligned}
& L \cdot \int_{(1-1/L)\cdot\xi_2}^{\xi_2} \Big(1 - h(\xi_1,\xi)\,d\xi\Big) \\
&= \xi_2 - L \cdot \int_{(1-1/L)\cdot\xi_2}^{\xi_2} h(\xi_1,\xi)\,d\xi \\
&= \xi_2 - L \cdot \int_{(1-1/L)\cdot\xi_2}^{\xi_1/(1-1/L)^k} h(\xi_1,\xi)\,d\xi - L \cdot \int_{\xi_1/(1-1/L)^k}^{\xi_2} h(\xi_1,\xi)\,d\xi \\
&= \xi_2 - L \cdot \int_{(1-1/L)\cdot\xi_2}^{\xi_1/(1-1/L)^k} \Big(1-\frac{1}{L}\Big)^k d\xi - L \cdot \int_{\xi_1/(1-1/L)^k}^{\xi_2} \Big(1-\frac{1}{L}\Big)^{k+1} d\xi \\
&= \xi_2 - L\cdot(\xi_1 - \xi_2\cdot(1-\tfrac{1}{L})^{k+1}) - L\cdot(\xi_2\cdot(1-\tfrac{1}{L})^{k+1} - \xi_1\cdot(1-\tfrac{1}{L})) \\
&= \xi_2 - \xi_1 \\
&= \max(\xi_2-\xi_1,0).
\end{aligned}$$

(2) Let $\xi_1 \in [0,1]$. We may assume that $\xi_1 > 0$, for otherwise

$$\int_{\xi_1}^{1} h(\xi_1, \xi)\, d\xi = \int_{\xi_1}^{1} 0\, d\xi = 0 \leq \left(1 - \frac{1}{L}\right)^L.$$

Let $k \in \{0, 1, 2, \dots\}$ such that $(1 - \frac{1}{L})^{k+1} < \xi_1 \leq (1 - \frac{1}{L})^k$. Then

$$\begin{aligned}
\int_{\xi_1}^{1} h(\xi_1, \xi)\, d\xi &= \int_{\xi_1/(1-1/L)^k}^{1} h(\xi_1, \xi)\, d\xi + \sum_{0 \leq k' < k} \int_{\xi_1/(1-1/L)^{k'}}^{\xi_1/(1-1/L)^{k'+1}} h(\xi_1, \xi)\, d\xi \\
&= \int_{\xi_1/(1-1/L)^k}^{1} \left(1 - \frac{1}{L}\right)^{k+1} d\xi + \sum_{0 \leq k' < k} \int_{\xi_1/(1-1/L)^{k'}}^{\xi_1/(1-1/L)^{k'+1}} \left(1 - \frac{1}{L}\right)^{k'+1} d\xi \\
&= \left(\left(1 - \frac{1}{L}\right)^{k+1} - \xi_1 \cdot \left(1 - \frac{1}{L}\right)\right) + \sum_{0 \leq k' < k} \frac{\xi_1}{L} \\
&= (1 - \tfrac{1}{L})^{k+1} + \tfrac{\xi_1}{L}(k - L + 1). \qquad (4.7)
\end{aligned}$$

We consider three cases.

Case 1: $k = L - 1$. Then (4.7) implies

$$\int_{\xi_1}^{1} h(\xi_1, \xi)\, d\xi = (1 - \tfrac{1}{L})^{k+1} + \tfrac{\xi_1}{L}(k - L + 1) = (1 - \tfrac{1}{L})^L.$$

Case 2: $k \geq L$. Then (4.7) implies

$$\begin{aligned}
\int_{\xi_1}^{1} h(\xi_1, \xi)\, d\xi &= (1 - \tfrac{1}{L})^{k+1} + \tfrac{\xi_1}{L}(k - L + 1) \\
&\leq (1 - \tfrac{1}{L})^{k+1} + \tfrac{1}{L}(k - L + 1)(1 - \tfrac{1}{L})^k \\
&= (1 - \tfrac{1}{L})^L \cdot \tfrac{k}{L} \cdot (1 - \tfrac{1}{L})^{k-L} \\
&\leq (1 - \tfrac{1}{L})^L \cdot e^{k/L-1} \cdot e^{(L-k)/L} \\
&= (1 - \tfrac{1}{L})^L,
\end{aligned}$$

where the first inequality follows from $\xi_1 \leq (1 - \frac{1}{L})^k$, and the second inequality follows from $e^{k/L-1} \geq \frac{k}{L}$ and $e^{-1/L} \geq 1 - \frac{1}{L}$.

Case 3: $k \le L-2$. Then (4.7) implies

$$\begin{aligned}\int_{\xi_1}^{1} h(\xi_1,\xi)\,d\xi &= (1-\tfrac{1}{L})^{k+1} + \tfrac{\xi_1}{L}(k-L+1)\\ &< (1-\tfrac{1}{L})^{k+1} - \tfrac{1}{L}(L-k-1)(1-\tfrac{1}{L})^{k+1}\\ &= (1-\tfrac{1}{L})^{L}\cdot \tfrac{k+1}{L-1}\cdot(1+\tfrac{1}{L-1})^{L-k-2}\\ &\le (1-\tfrac{1}{L})^{L}\cdot e^{(k+1)/(L-1)-1}\cdot e^{(L-k-2)/(L-1)}\\ &= (1-\tfrac{1}{L})^{L},\end{aligned}$$

where the first inequality follows from $\xi_1 > (1-\frac{1}{L})^{k+1}$, and the second inequality follows from $e^{(k+1)/(L-1)-1} \ge \frac{k+1}{L-1}$ and $e^{1/(L-1)} \ge 1+\frac{1}{L-1}$. $\square$

Lemma 4.27 below is obtained by combining Lemmas 4.22, 4.25, and 4.26. Our main results are presented in Theorem 4.28 and 4.29 and proved using Lemma 4.27.

Lemma 4.27. $\displaystyle\sum_{(i,j)\in I\times J} x_{i,j} \le \left(1+\left(1-\frac{1}{L}\right)^{L}\right)\cdot|\mu|.$

Proof. Consider the charging argument with the exchange function h as defined by (4.6). By part (1) of Lemma 4.26, the function h satisfies (H1)–(H4). Lemma 4.22 implies

$$\begin{aligned}\sum_{(i,j)\in I\times J} x_{i,j} &\le \sum_{(i,j)\in I\times J}(\theta_{i,j}+\phi_{i,j})\\ &= \sum_{(i,j)\in\mu}\Big(\sum_{j'\in J}\theta_{i,j'}+\sum_{i'\in I}\phi_{i',j}\Big) + \sum_{\substack{i\in I\\ \mu(i)=0}}\sum_{j\in J}\theta_{i,j} + \sum_{\substack{j\in J\\ \mu(j)=0}}\sum_{i\in I}\phi_{i,j}. \qquad (4.8)\end{aligned}$$

Part (1) of Lemma 4.25 implies

$$
\begin{aligned}
\sum_{(i,j)\in\mu}\Big(\sum_{j'\in J}\theta_{i,j'}+\sum_{i'\in I}\phi_{i',j}\Big) &\le \sum_{(i,j)\in\mu}\Big(1+\int_{1-p_i}^{1} h(1-p_i,\xi)\,d\xi\Big)\\
&\le \sum_{(i,j)\in\mu}\Big(1+\Big(1-\frac{1}{L}\Big)^{L}\Big)\\
&= (1+(1-\tfrac{1}{L})^{L})\cdot|\mu|, \qquad (4.9)
\end{aligned}
$$

where the second inequality follows from part (2) of Lemma 4.26. Part (2) of Lemma 4.25 implies

$$\sum_{\substack{i\in I\\ \mu(i)=0}}\sum_{j\in J}\theta_{i,j}=0. \tag{4.10}$$

The definitions of $\{\phi_{i,j}\}_{(i,j)\in I\times J}$ imply

$$\sum_{\substack{j\in J\\ \mu(j)=0}}\sum_{i\in I}\phi_{i,j}=0. \tag{4.11}$$

Combining (4.8)–(4.11) gives the desired inequality. □

Theorem 4.28. *For the maximum stable matching problem with one-sided ties and tie length at most L, Algorithms 4.1 and 4.2 are polynomial-time $(1+(1-\frac{1}{L})^{L})$-approximation algorithms.*

Proof. Algorithms 4.1 and 4.2 each run in polynomial time because linear programming is polynomial-time solvable and the number of iterations of the loop is at most $|I|\times|J|$.

By Lemma 4.13 and 4.21, Algorithms 4.1 and 4.2 each produce a matching μ such that $(\mu,\mathbf{p})$ satisfies (P1)–(P4) with $\eta=0$. So Lemma 4.14 implies that the output μ is a weakly stable matching. Let μ' be a maximum weakly stable matching, and $\mathbf{x}'$ be the indicator variables of μ'. Since μ' is weakly stable, $\mathbf{x}'$ satisfies constraints (C1)–(C5). Hence Lemma 4.27 implies

$$\left(1 + \left(1 - \frac{1}{L}\right)^L\right) \cdot |\mu| \geq \sum_{(i,j) \in I \times J} x_{i,j} \geq \sum_{(i,j) \in I \times J} x'_{i,j} = |\mu'|,$$

where the second inequality follows from the optimality of $\mathbf{x}$. $\square$

Theorem 4.29. *For the maximum stable matching problem with one-sided ties and tie length at most L, the integrality gap of the linear programming formulation in Section 4.4 is $1 + (1 - \frac{1}{L})^L$.*

Proof. Consider the matching μ produced by Algorithms 4.1 or 4.2. By Lemma 4.13 and 4.21, there exists $\mathbf{p}$ such that $(\mu, \mathbf{p})$ satisfies (P1)–(P4) with $\eta = 0$. So Lemma 4.14 implies that the output μ is a weakly stable matching. Hence Lemma 4.27 implies

$$\left(1 + \left(1 - \frac{1}{L}\right)^L\right) \cdot |\mu| \geq \sum_{(i,j) \in I \times J} x_{i,j}.$$

Let $\mathbf{x}'$ be the indicator variables of μ. Since μ is weakly stable, Lemma 4.1 implies that $\mathbf{x}'$ is an integral solution satisfying constraints (C1)–(C5). Since

$$\left(1 + \left(1 - \frac{1}{L}\right)^L\right) \cdot \sum_{(i,j) \in I \times J} x'_{i,j} = \left(1 + \left(1 - \frac{1}{L}\right)^L\right) \cdot |\mu| \geq \sum_{i \in I} \sum_{j \in J} x_{i,j},$$

the integrality gap is at most $1 + (1 - \frac{1}{L})^L$. This upper bound matches the known lower bound for the integrality gap [52, Section 5.1]. $\square$

Chapter 5

Concluding Remarks

We conclude this dissertation by summarizing our results and mentioning some potential future work.

5.1 Group Strategyproof Pareto-Stable Mechanism

In Chapter 3, we show the existence of a group strategyproof Pareto-stable mechanism for stable matching markets with indifferences. We achieve this by drawing a connection between the stable marriage market and the generalized assignment game. It would be interesting to study this connection closely to discover more similarities between these two models. Even though we show that every stable outcome of the associated generalized assignment game is Pareto-stable in the stable marriage market with indifferences, it is unclear whether every Pareto-stable matching in a stable marriage market with indifferences appears as some stable outcome of some associated generalized assignment game. Also, even though the Gale-Shapley algorithm is the unique stable marriage mechanism that is strategyproof for the men when preferences are strict, our mechanisms suggest that when preferences are weak, there are multiple Pareto-stable marriage mechanisms that are strategyproof for the men, since there are multiple ways to convert ordinal preferences into cardinal utilities. It remains open to characterize all such mechanisms. Another possible direction is to study the effects of the different ways in which ordinal preferences can be converted into cardinal utilities.

Extension of our results to more general models can be also considered. Since some strategyproofness results are known for the models of matching with contracts [41] and matching in networks [42] when preferences are strict, it would be interesting to know whether our results can be extended to these models.

5.2 Maximum Stable Matchings

In Chapter 4, we present a polynomial-time algorithm for the maximum stable matching problem that achieves an approximation ratio of $1 + (1 - \frac{1}{L})^L$ for the case of one-sided ties where the tie length is at most L. When $L = 2$, this gives an approximation ratio and integrality gap of $\frac{5}{4}$, matching the known UG-hardness result [39]. For $L > 2$, it remains open whether better hardness results can be obtained.

For the case of two-sided ties where the length of each tie is at most L, it is known that the integrality gap is at least $\frac{3L-2}{2L-1}$. A natural question is whether our algorithm can be generalized to handle such cases and achieve an approximation ratio that matches the integrality gap. Generalizing our algorithm to the case of two-sided ties seems to require a better understanding of the linear programming formulation and its relationship with the assignment game. Perhaps we can reformulate the proposal process with priorities as a generalized assignment game, where the use of the numerical priorities as tie-breakers is modeled as men paying women monetary compensation to break the ties. For the case where both the men and the women are allowed to have ties in their preferences, a potential path towards extending our algorithm is to treat the two sides of the market symmetrically and to allow both men and women to pay monetary compensation to their partner for tie-breaking purposes.

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