Section 2.5 - Multiplying Partitioned Matrices

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Theorem

Let $C \in \mathbb{R}^{m \times n}$, $A \in \mathbb{R}^{m \times k}$, and $B \in \mathbb{R}^{k \times n}$. Partition (conformally)

$$C = \left(\begin{array}{c|c|c|c} C_{0,0} & C_{0,1} & \cdots & C_{0,N-1} \\ \hline C_{1,0} & C_{1,1} & \cdots & C_{1,N-1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline C_{M-1,0} & C_{M-1,1} & \cdots & C_{M-1,N-1} \end{array} \right),$$

$$A = \left(\begin{array}{c|c|c|c} A_{0,0} & A_{0,1} & \cdots & A_{0,K-1} \\ \hline A_{1,0} & A_{1,1} & \cdots & A_{1,K-1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline A_{M-1,0} & A_{M-1,1} & \cdots & A_{M-1,K-1} \end{array} \right),$$

$$B = \left(\begin{array}{c|c|c|c} B_{0,0} & B_{0,1} & \cdots & B_{0,N-1} \\ \hline B_{1,0} & B_{1,1} & \cdots & B_{1,N-1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline B_{K-1,0} & B_{K-1,1} & \cdots & B_{K-1,N-1} \end{array} \right).$$

Then $C_{i,j} = \sum_{p=0}^{K-1} A_{i,p} B_{p,j}$.

Note

- **If** one partitions matrices C, A, and B into blocks,
- **and** one makes sure the dimensions match up,
- **then** blocked matrix-matrix multiplication proceeds exactly as does a regular matrix-matrix multiplication
- **except** that individual multiplications of scalars commute while (in general) individual multiplications with matrix blocks (submatrices) do not.

Example

Consider

$$A = \left(\begin{array}{rr|rr} -1 & 2 & 4 & 1 \\ 1 & 0 & -1 & -2 \\ 2 & -1 & 3 & 1 \\ 1 & 2 & 3 & 4 \end{array}\right), \quad B = \left(\begin{array}{rrr} -2 & 2 & -3 \\ 0 & 1 & -1 \\ \hline -2 & -1 & 0 \\ 4 & 0 & 1 \end{array}\right),$$

then

$$AB = \left(\begin{array}{rrr} -2 & -4 & 2 \\ -8 & 3 & -5 \\ -6 & 0 & -4 \\ 8 & 1 & -1 \end{array}\right).$$

Example (continued)

$$
\underbrace{\left(\begin{array}{rr|rr} -1 & 2 & 4 & 1 \\ 1 & 0 & -1 & -2 \\ 2 & -1 & 3 & 1 \\ 1 & 2 & 3 & 4 \end{array}\right)}_{A}
\underbrace{\left(\begin{array}{rrr} -2 & 2 & -3 \\ 0 & 1 & -1 \\ \hline -2 & -1 & 0 \\ 4 & 0 & 1 \end{array}\right)}_{B}
$$

$$
= \underbrace{\begin{pmatrix} -1 & 2 \\ 1 & 0 \\ 2 & -1 \\ 1 & 2 \end{pmatrix}}_{A_0}
\underbrace{\begin{pmatrix} -2 & 2 & -3 \\ 0 & 1 & -1 \end{pmatrix}}_{B_0}
+ \underbrace{\begin{pmatrix} 4 & 1 \\ -1 & -2 \\ 3 & 1 \\ 3 & 4 \end{pmatrix}}_{A_1}
\underbrace{\begin{pmatrix} -2 & -1 & 0 \\ 4 & 0 & 1 \end{pmatrix}}_{B_1}
$$

$$
= \underbrace{\begin{pmatrix} 2 & 0 & 1 \\ -2 & 2 & -3 \\ -4 & 3 & -5 \\ -2 & 4 & -5 \end{pmatrix}}_{A_0 B_0}
+ \underbrace{\begin{pmatrix} -4 & -4 & 1 \\ -6 & 1 & -2 \\ -2 & -3 & 1 \\ 10 & -3 & 4 \end{pmatrix}}_{A_1 B_1}
= \underbrace{\begin{pmatrix} -2 & -4 & 2 \\ -8 & 3 & -5 \\ -6 & 0 & -4 \\ 8 & 1 & -1 \end{pmatrix}}_{AB}.
$$

Corollary

Partition C and B by columns and do not partition A. Then

$$C = \left(\begin{array}{c|c|c|c} c_0 & c_1 & \cdots & c_{n-1} \end{array} \right) \quad \text{and} \quad B = \left(\begin{array}{c|c|c|c} b_0 & b_1 & \cdots & b_{n-1} \end{array} \right)$$

so that

$$\begin{array}{rcl} \left(\begin{array}{c|c|c|c} c_0 & c_1 & \cdots & c_{n-1} \end{array} \right) & = & C = AB = A \left(\begin{array}{c|c|c|c} b_0 & b_1 & \cdots & b_{n-1} \end{array} \right) \\ & = & \left(\begin{array}{c|c|c|c} Ab_0 & Ab_1 & \cdots & Ab_{n-1} \end{array} \right). \end{array}$$

Example

$$\begin{pmatrix} -1 & 2 & 4 \\ 1 & 0 & -1 \\ 2 & -1 & 3 \end{pmatrix} \left(\begin{array}{r|r} -2 & 2 \\ 0 & 1 \\ -2 & -1 \end{array}\right)$$

$$= \left(\begin{pmatrix} -1 & 2 & 4 \\ 1 & 0 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} \middle| \begin{pmatrix} -1 & 2 & 4 \\ 1 & 0 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right)$$

$$= \left(\begin{array}{r|r} -6 & -4 \\ 0 & 3 \\ -10 & 0 \end{array}\right)$$

By moving the loop indexed by j to the outside in the algorithm for computing $C = AB + C$ we observe that

for $j = 0, \ldots, n-1$
  for $i = 0, \ldots, m-1$
    for $p = 0, \ldots, k-1$
      $\gamma_{i,j} := \alpha_{i,p}\beta_{p,j} + \gamma_{i,j}$
    endfor
  endfor
endfor

$\left.\right\} c_j := Ab_j + c_j$ (the loops over i and p)

or

for $j = 0, \ldots, n-1$
  for $p = 0, \ldots, k-1$
    for $i = 0, \ldots, m-1$
      $\gamma_{i,j} := \alpha_{i,p}\beta_{p,j} + \gamma_{i,j}$
    endfor
  endfor
endfor

$\left.\right\} c_j := Ab_j + c_j$ (the loops over p and i)

Corollary

Partition C and A by rows and do not partition B. Then

$$C = \left(\begin{array}{c} \tilde{c}_0^T \\ \hline \tilde{c}_1^T \\ \hline \vdots \\ \hline \tilde{c}_{m-1}^T \end{array} \right) \quad \text{and} \quad A = \left(\begin{array}{c} \tilde{a}_0^T \\ \hline \tilde{a}_1^T \\ \hline \vdots \\ \hline \tilde{a}_{m-1}^T \end{array} \right)$$

so that

$$\left(\begin{array}{c} \tilde{c}_0^T \\ \hline \tilde{c}_1^T \\ \hline \vdots \\ \hline \tilde{c}_{m-1}^T \end{array} \right) = C = AB = \left(\begin{array}{c} \tilde{a}_0^T \\ \hline \tilde{a}_1^T \\ \hline \vdots \\ \hline \tilde{a}_{m-1}^T \end{array} \right) B = \left(\begin{array}{c} \tilde{a}_0^T B \\ \hline \tilde{a}_1^T B \\ \hline \vdots \\ \hline \tilde{a}_{m-1}^T B \end{array} \right).$$

Example

$$\left(\begin{array}{rrr} -1 & 2 & 4 \\ \hline 1 & 0 & -1 \\ \hline 2 & -1 & 3 \end{array}\right)\left(\begin{array}{rr} -2 & 2 \\ 0 & 1 \\ -2 & -1 \end{array}\right)$$

$$= \left(\begin{array}{c} \left(\begin{array}{rrr} -1 & 2 & 4 \end{array}\right)\left(\begin{array}{rr} -2 & 2 \\ 0 & 1 \\ -2 & -1 \end{array}\right) \\ \hline \left(\begin{array}{rrr} 1 & 0 & -1 \end{array}\right)\left(\begin{array}{rr} -2 & 2 \\ 0 & 1 \\ -2 & -1 \end{array}\right) \\ \hline \left(\begin{array}{rrr} 2 & -1 & 3 \end{array}\right)\left(\begin{array}{rr} -2 & 2 \\ 0 & 1 \\ -2 & -1 \end{array}\right) \end{array}\right) = \left(\begin{array}{rr} -6 & -4 \\ \hline 0 & 3 \\ \hline -10 & 0 \end{array}\right)$$

In the algorithm for computing $C = AB + C$ the loop indexed by i can be moved to the outside so that

$$\left.\begin{array}{l}\textbf{for } i = 0, \ldots, m-1 \\ \quad \textbf{for } j = 0, \ldots, n-1 \\ \quad\quad \textbf{for } p = 0, \ldots, k-1 \\ \quad\quad\quad \gamma_{i,j} := \alpha_{i,p}\beta_{p,j} + \gamma_{i,j} \\ \quad\quad \textbf{endfor} \\ \quad \textbf{endfor} \\ \textbf{endfor}\end{array}\right\} \tilde{c}_i^T := \tilde{a}_i^T B + \tilde{c}_i^T$$

or

$$\left.\begin{array}{l}\textbf{for } i = 0, \ldots, m-1 \\ \quad \textbf{for } p = 0, \ldots, k-1 \\ \quad\quad \textbf{for } j = 0, \ldots, n-1 \\ \quad\quad\quad \gamma_{i,j} := \alpha_{i,p}\beta_{p,j} + \gamma_{i,j} \\ \quad\quad \textbf{endfor} \\ \quad \textbf{endfor} \\ \textbf{endfor}\end{array}\right\} \tilde{c}_i^T := \tilde{a}_i^T B + \tilde{c}_i^T$$

Corollary

Partition A and B by columns and rows, respectively, and do not partition C. Then

$$A = \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{k-1} \end{array} \right) \quad \text{and} \quad B = \left(\begin{array}{c} \tilde{b}_0^T \\ \hline \tilde{b}_1^T \\ \hline \vdots \\ \hline \tilde{b}_{k-1}^T \end{array} \right)$$

so that

$$\begin{aligned} C &= AB = \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{k-1} \end{array} \right) \left(\begin{array}{c} \tilde{b}_0^T \\ \hline \tilde{b}_1^T \\ \hline \vdots \\ \hline \tilde{b}_{k-1}^T \end{array} \right) \\ &= a_0 \tilde{b}_0^T + a_1 \tilde{b}_1^T + \cdots + a_{k-1} \tilde{b}_{k-1}^T. \end{aligned}$$

Example

$$\left(\begin{array}{r|r|r} -1 & 2 & 4 \\ 1 & 0 & -1 \\ 2 & -1 & 3 \end{array}\right)\left(\begin{array}{rr} -2 & 2 \\ \hline 0 & 1 \\ \hline -2 & -1 \end{array}\right)$$

$$= \left(\begin{array}{r} -1 \\ 1 \\ 2 \end{array}\right)\left(\begin{array}{rr} -2 & 2 \end{array}\right) + \left(\begin{array}{r} 2 \\ 0 \\ -1 \end{array}\right)\left(\begin{array}{rr} 0 & 1 \end{array}\right) + \left(\begin{array}{r} 4 \\ -1 \\ 3 \end{array}\right)\left(\begin{array}{rr} -2 & -1 \end{array}\right)$$

$$= \left(\begin{array}{rr} 2 & -2 \\ -2 & 2 \\ -4 & 4 \end{array}\right) + \left(\begin{array}{rr} 0 & 2 \\ 0 & 0 \\ 0 & -1 \end{array}\right) + \left(\begin{array}{rr} -8 & -4 \\ 2 & 1 \\ -6 & -3 \end{array}\right) = \left(\begin{array}{rr} -6 & -4 \\ 0 & 3 \\ -10 & 0 \end{array}\right)$$

In the algorithm for computing $C = AB + C$ the loop indexed by p can be moved to the outside so that

$$
\begin{array}{l}
\textbf{for } p = 0, \ldots, k-1 \\
\quad \left.\begin{array}{l}
\textbf{for } j = 0, \ldots, n-1 \\
\quad \textbf{for } i = 0, \ldots, m-1 \\
\quad\quad \gamma_{i,j} := \alpha_{i,p}\beta_{p,j} + \gamma_{i,j} \\
\quad \textbf{endfor} \\
\textbf{endfor}
\end{array}\right\} C := a_p \tilde{b}_p^T + C \\
\textbf{endfor}
\end{array}
$$

or

$$
\begin{array}{l}
\textbf{for } p = 0, \ldots, k-1 \\
\quad \left.\begin{array}{l}
\textbf{for } i = 0, \ldots, m-1 \\
\quad \textbf{for } j = 0, \ldots, n-1 \\
\quad\quad \gamma_{i,j} := \alpha_{i,p}\beta_{p,j} + \gamma_{i,j} \\
\quad \textbf{endfor} \\
\textbf{endfor}
\end{array}\right\} C := a_p \tilde{b}_p^T + C \\
\textbf{endfor}
\end{array}
$$

Example

Partition C into elements (scalars) and A and B by rows and columns, respectively, and do not partition C. Then

$$
\begin{array}{rcl}
C & = & \left(\begin{array}{c|c|c|c}
\gamma_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,n-1} \\ \hline
\gamma_{1,0} & \gamma_{1,1} & \cdots & \gamma_{1,n-1} \\ \hline
\vdots & \vdots & \ddots & \vdots \\ \hline
\gamma_{m-1,0} & \gamma_{m-1,1} & \cdots & \gamma_{m-1,n-1}
\end{array}\right) \\
& = & \left(\begin{array}{c}
\tilde{a}_0^T \\ \hline
\tilde{a}_1^T \\ \hline
\vdots \\ \hline
\tilde{a}_{m-1}^T
\end{array}\right)
\left(\begin{array}{c|c|c|c}
b_0 & b_1 & \cdots & b_{n-1}
\end{array}\right) \\
& = & \left(\begin{array}{c|c|c|c}
\tilde{a}_0^T b_0 & \tilde{a}_0^T b_1 & \cdots & \tilde{a}_0^T b_{n-1} \\ \hline
\tilde{a}_1^T b_0 & \tilde{a}_1^T b_1 & \cdots & \tilde{a}_1^T b_{n-1} \\ \hline
\vdots & \vdots & \ddots & \vdots \\ \hline
\tilde{a}_{m-1}^T b_0 & \tilde{a}_{m-1}^T b_1 & \cdots & \tilde{a}_{m-1}^T b_{n-1}
\end{array}\right).
\end{array}
$$

As expected, $\gamma_{i,j} = \tilde{a}_i^T b_j$: the dot product of the ith row of A with the jth row of B.

Example

$$\begin{pmatrix} -1 & 2 & 4 \\ 1 & 0 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ 0 & 1 \\ -2 & -1 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} \begin{pmatrix} -1 & 2 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} & \begin{pmatrix} -1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \\ \hline \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} & \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \\ \hline \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} & \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \end{array}\right)$$

$$= \begin{pmatrix} -6 & -4 \\ 0 & 3 \\ -10 & 0 \end{pmatrix}$$

In the algorithm for computing $C = AB + C$ the loop indexed by p (which computes the dot product of the ith row of A with the jth column of B) can be moved to the inside so that

for $j = 0, \ldots, n-1$
  for $i = 0, \ldots, m-1$
    for $p = 0, \ldots, k-1$
      $\gamma_{i,j} := \alpha_{i,p}\beta_{p,j} + \gamma_{i,j}$
    endfor
  endfor
endfor

(the innermost loop over p computes) $\gamma_{i,j} := \tilde{a}_i^T b_j + \gamma_{i,j}$

or

for $i = 0, \ldots, m-1$
  for $j = 0, \ldots, n-1$
    for $p = 0, \ldots, k-1$
      $\gamma_{i,j} := \alpha_{i,p}\beta_{p,j} + \gamma_{i,j}$
    endfor
  endfor
endfor

(the innermost loop over p computes) $\gamma_{i,j} := \tilde{a}_i^T b_j + \gamma_{i,j}$

Summing it all up

for $j = 0, \ldots, n-1$
  $c_j := Ab_j + c_j$
endfor

for $j = 0, \ldots, n-1$
  for $i = 0, \ldots, m-1$
    $\gamma_{i,j} := \check{a}_i^T b_j + \gamma_{i,j}$
  endfor
endfor

for j =
  for i =
    for p
      $\gamma_{i,j}$:
    endfo
  endfor
endfor

for $j = 0, \ldots, n-1$
  for $p = 0, \ldots, k-1$
    $c_j := \beta_{p,j} a_p + c_j$
  endfor
endfor

for j =
  for p =
    for i =
      $\gamma_{i,j}$:
    endfo
  endfor
endfor

for $i = 0, \ldots, m-1$
  for $j = 0, \ldots, n-1$
    $\gamma_{i,j} := \check{a}_i^T b_j + \gamma_{i,j}$

for i =
  for j =
    for p
      $\gamma_{i,j}$: