Enumerating Rationals Without Repetition

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- $\mathbb{P}$ is the **positive integers**.

- $\mathbb{N}$ is the **nonnegative integers**.

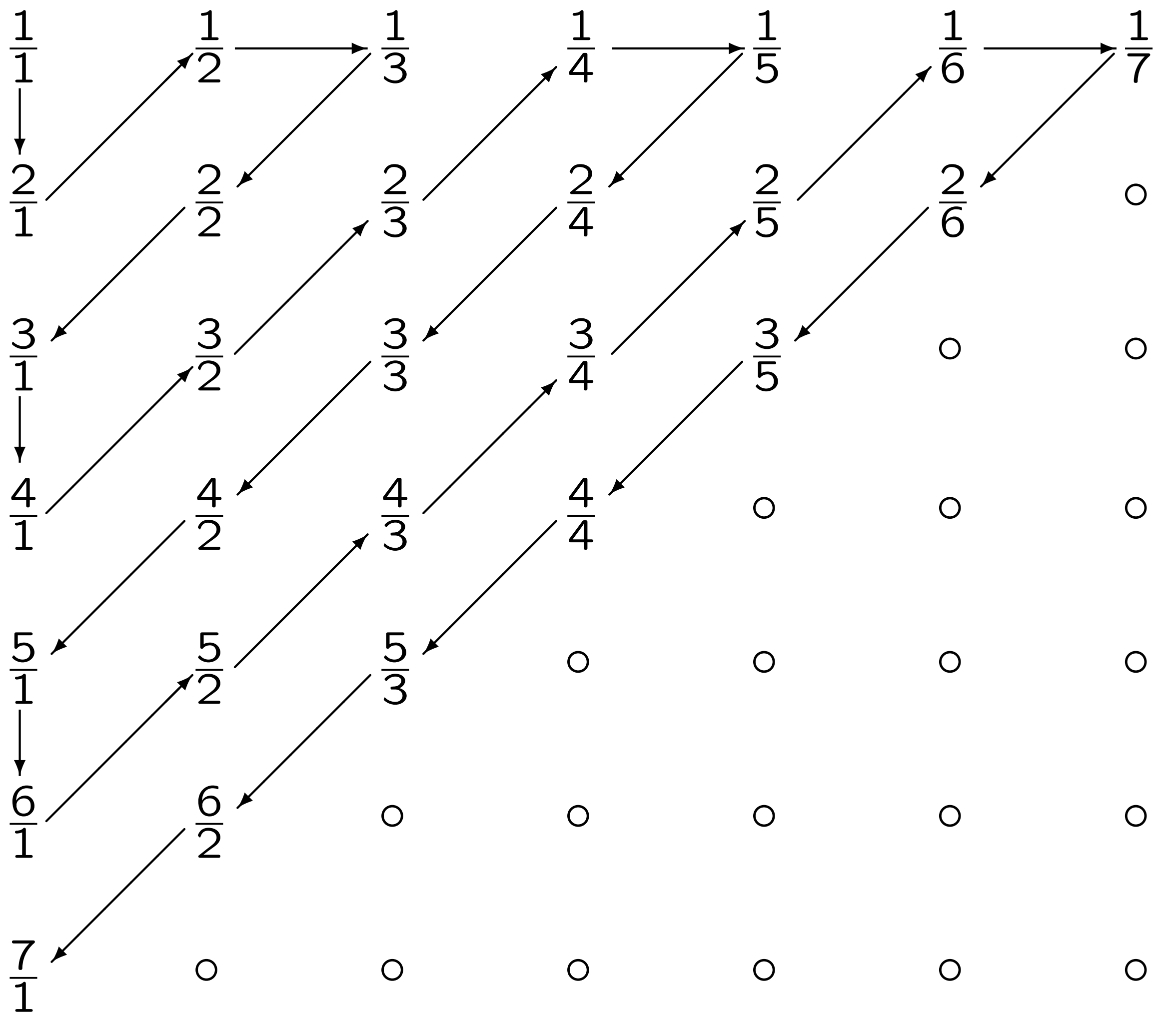
- $\mathbb{Q}^+$ is the **positive rationals**.

- S is an infinite set.

- An **enumeration** of S is a function F from $\mathbb{P}$ (or $\mathbb{N}$) **onto** S.

- An **enumeration without repetition** of S is an enumeration of S that is also **one to one**.

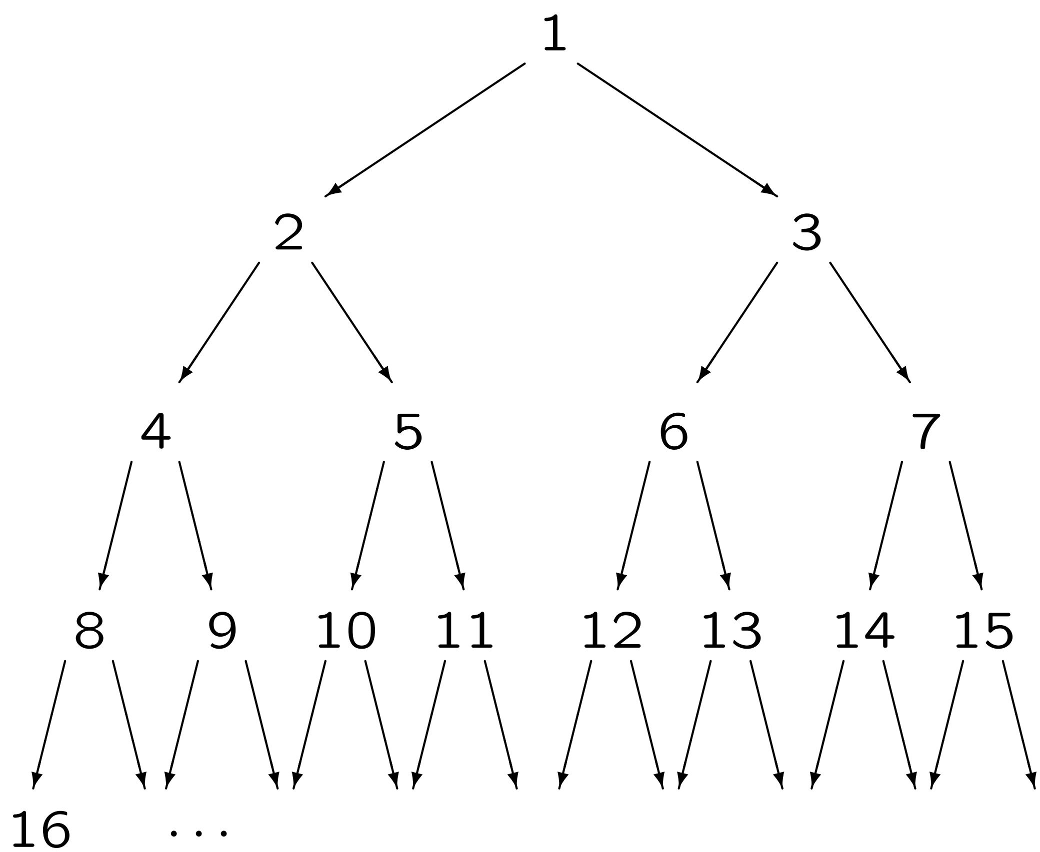
An **enumeration** of the positive rationals (with repetition)



$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{4}, \frac{3}{3}, \frac{4}{2}, \frac{5}{1},$$

$$\frac{6}{1}, \frac{5}{2}, \frac{4}{3}, \frac{3}{4}, \frac{2}{5}, \frac{1}{6}, \frac{1}{7}, \frac{2}{6}, \frac{3}{5}, \frac{4}{4}, \frac{5}{3}, \frac{6}{2}, \frac{7}{1}, \ldots$$

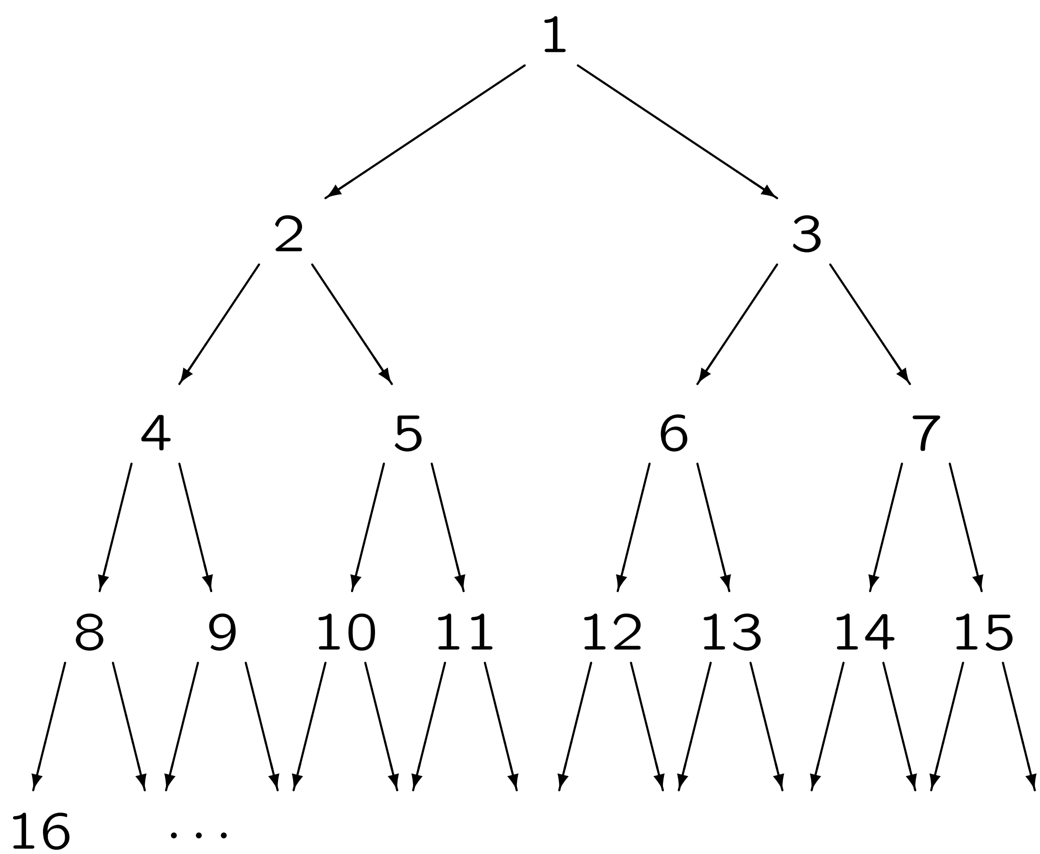
An **enumeration** of the nodes
of an infinite binary tree
(without repetition)



root: 1

left children: Even integers > 0

right children: Odd integers > 1



Parent to left child: $n \mapsto 2 \cdot n$

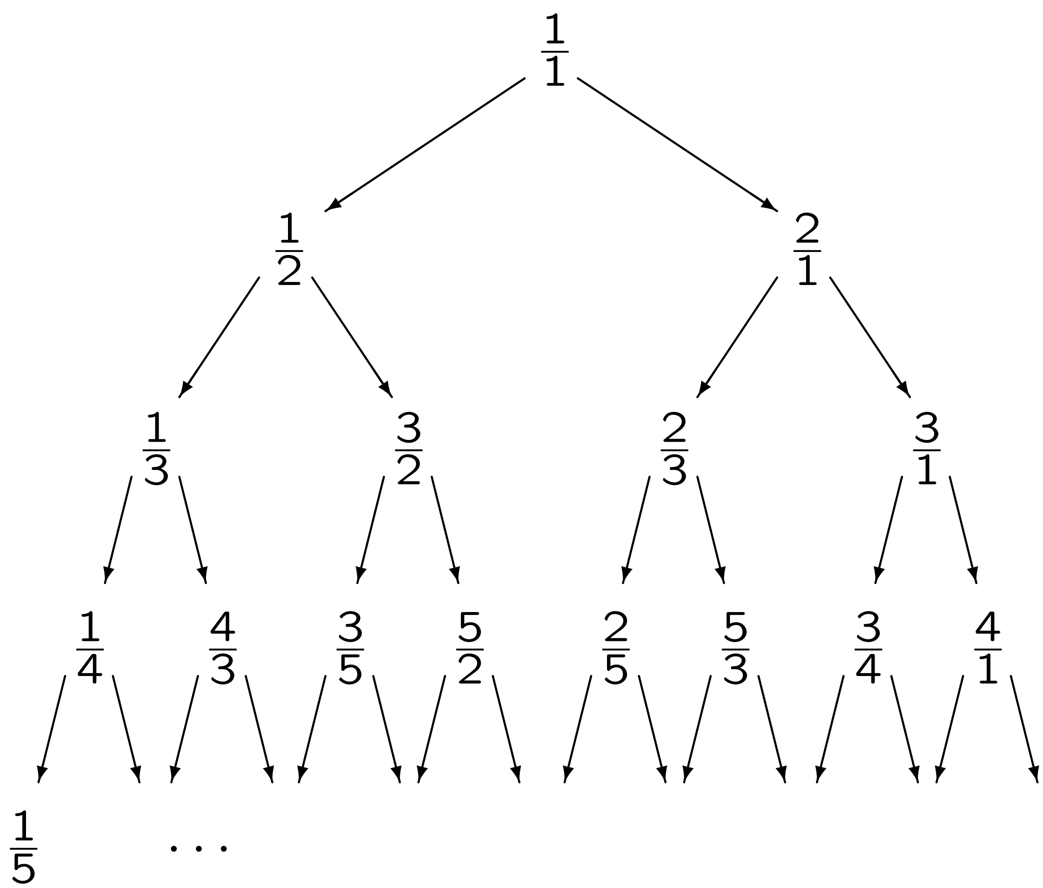
Parent to right child: $n \mapsto 2 \cdot n + 1$

Left child to parent: $n \mapsto \frac{n}{2}$

Right child to parent: $n \mapsto \frac{n-1}{2}$

Parent $<$ Child

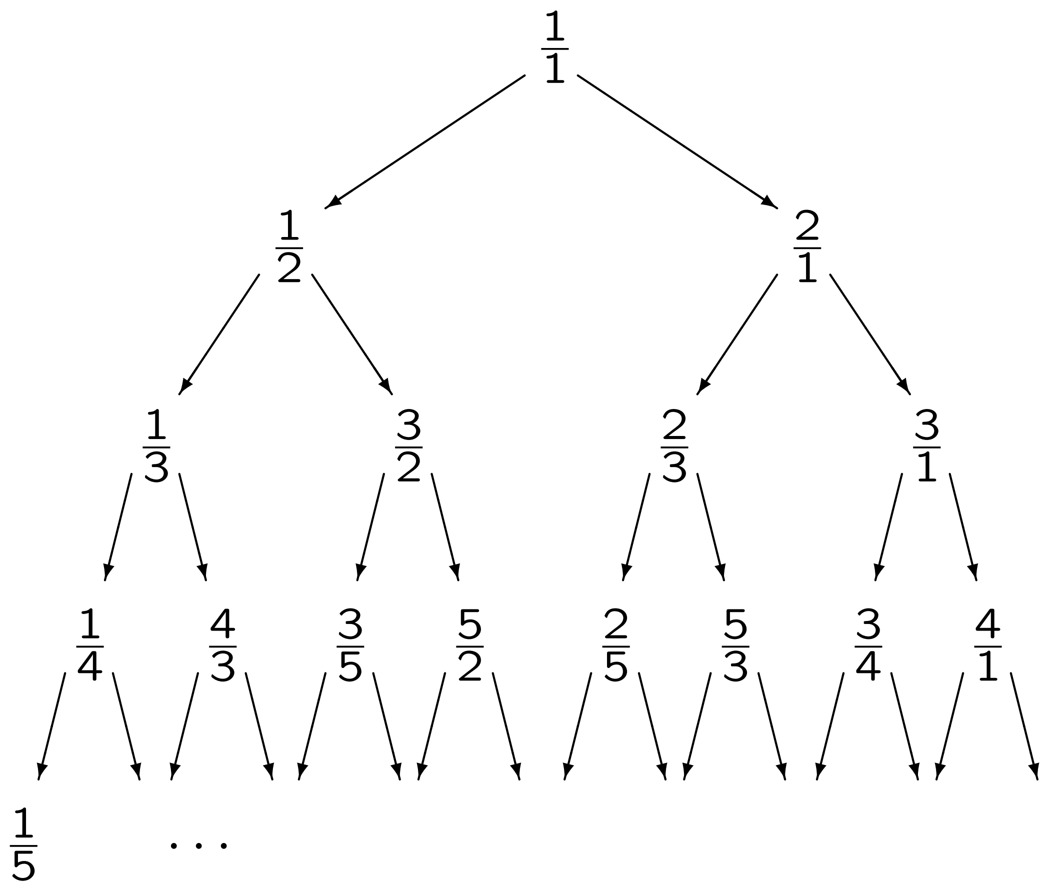
The **Calkin-Wilf** infinite binary tree



root: $\frac{1}{1}$

left children: Positive rationals < 1

right children: Rationals > 1



Parent to left child: $\frac{r}{s} \mapsto \frac{r}{r+s}$

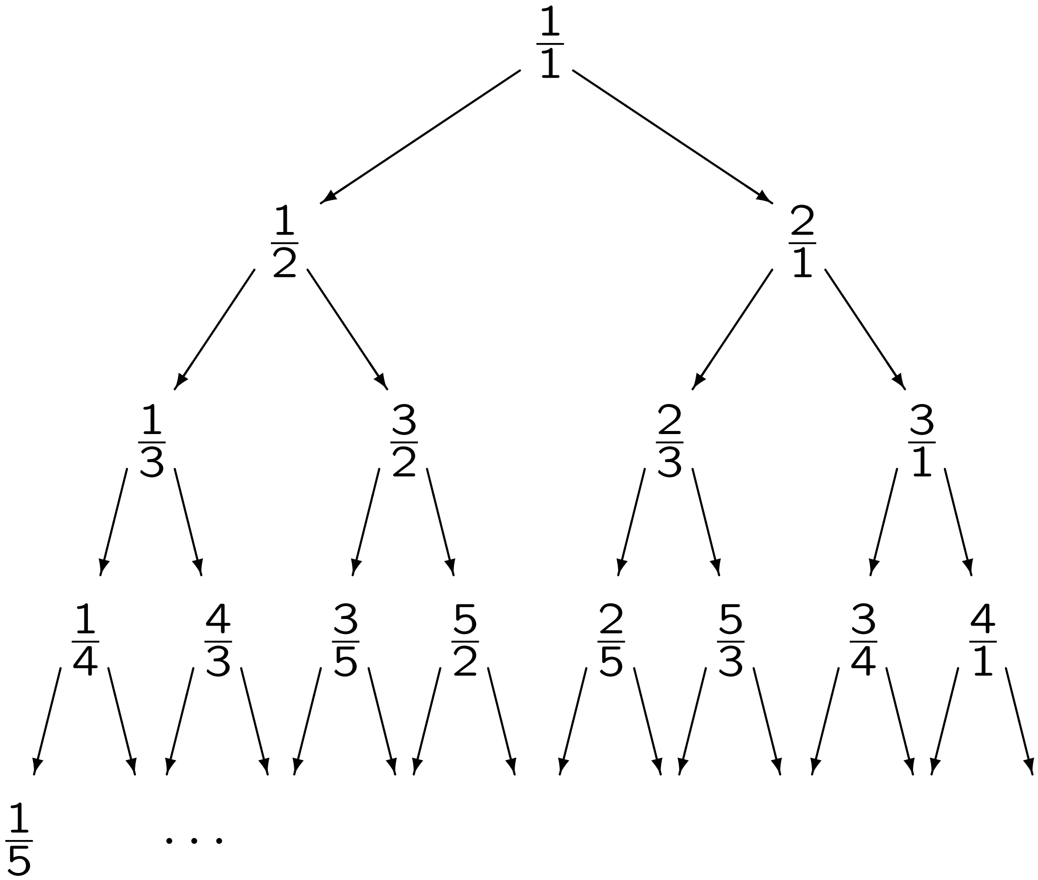
Parent to right child: $\frac{r}{s} \mapsto \frac{r+s}{s}$

Left child to parent: $\frac{r}{s} \mapsto \frac{r}{s-r}$

Right child to parent: $\frac{r}{s} \mapsto \frac{r-s}{s}$

Measure of node: $m\left(\frac{r}{s}\right) = r + s$

$$m(\textbf{Parent}) < m(\textbf{Child})$$



Parent to left child: $\frac{r}{s} \mapsto \frac{r}{r+s}$

Parent to right child: $\frac{r}{s} \mapsto \frac{r+s}{s}$

Each rational in tree is represented by a reduced fraction:

- $\text{gcd}(1, 1) = 1$
- $\text{gcd}(r + s, s) = \text{gcd}(r, s) = \text{gcd}(r, r + s)$

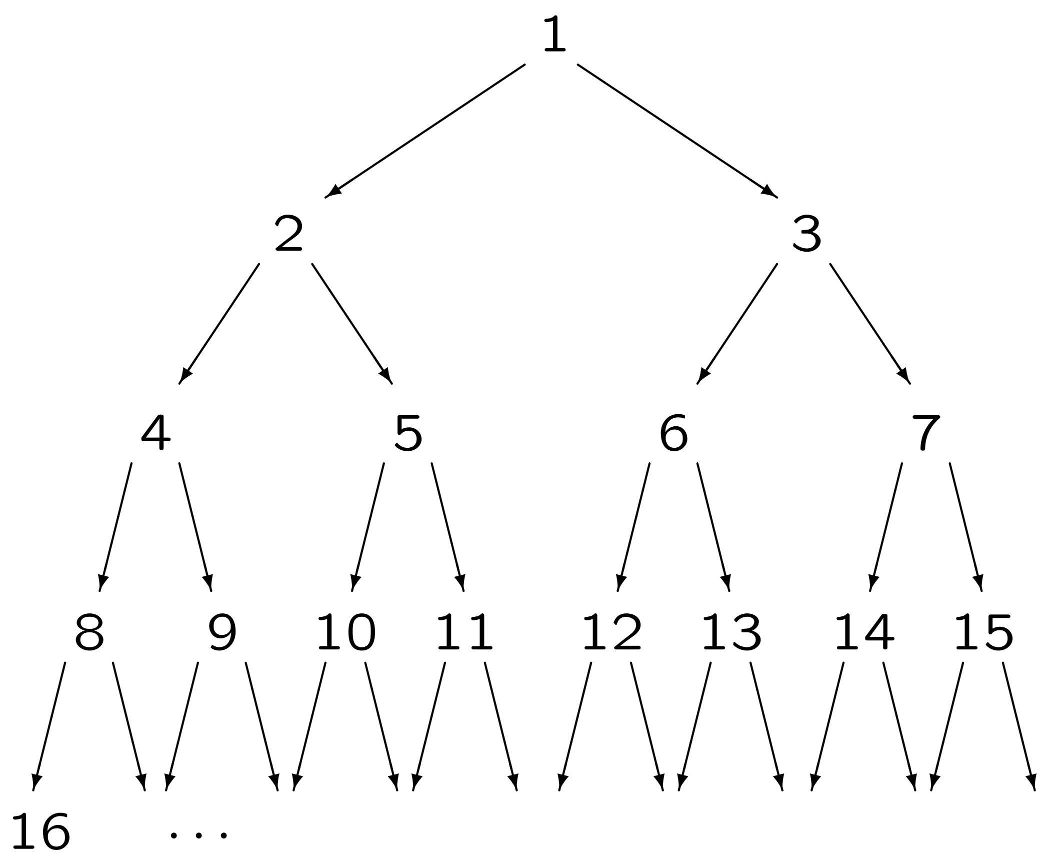
Identify the nodes in the two trees:

- Map root to root.

- If node is a left child then
  - Compute parent, P, in same tree
  - Recursively find node, P', in other tree, corresponding to P.
  - Return left child of P'.

- If node is a right child then
  - Compute parent, P, in same tree
  - Recursively find node, P', in other tree, corresponding to P.
  - Return right child of P'.

Obtain two functions: $f : \mathbb{P} \mapsto \mathbb{Q}^+$,
$g : \mathbb{Q}^+ \mapsto \mathbb{P}$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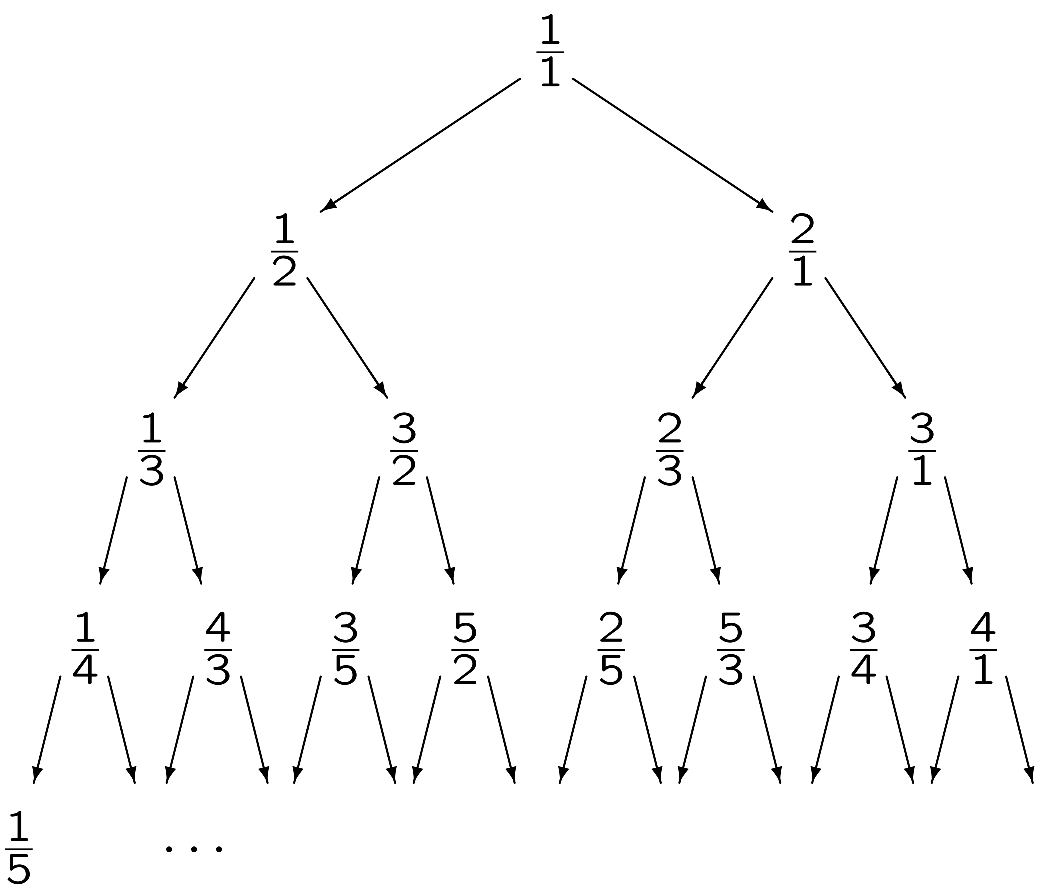
$\frac{1}{1}$
$\frac{1}{2}$ $\frac{2}{1}$
$\frac{1}{3}$ $\frac{3}{2}$ $\frac{2}{3}$ $\frac{3}{1}$
$\frac{1}{4}$ $\frac{4}{3}$ $\frac{3}{5}$ $\frac{5}{2}$ $\frac{2}{5}$ $\frac{5}{3}$ $\frac{3}{4}$ $\frac{4}{1}$
$\frac{1}{5}$ . . .

The two functions: $f : \mathbb{P} \mapsto \mathbb{Q}^+$,
$g : \mathbb{Q}^+ \mapsto \mathbb{P}$,
are functional inverses.

This means $f : \mathbb{P} \mapsto \mathbb{Q}^+$ is both **onto** and **one to one**.

So $f : \mathbb{P} \mapsto \mathbb{Q}^+$ is an enumeration **without repitition**:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}, \frac{1}{5}, \ldots$$

N. Calkin and H. Wilf, **Recounting the rationals**, Amer. Math. Monthly 107 (2000), 360–363.