Semantic Security

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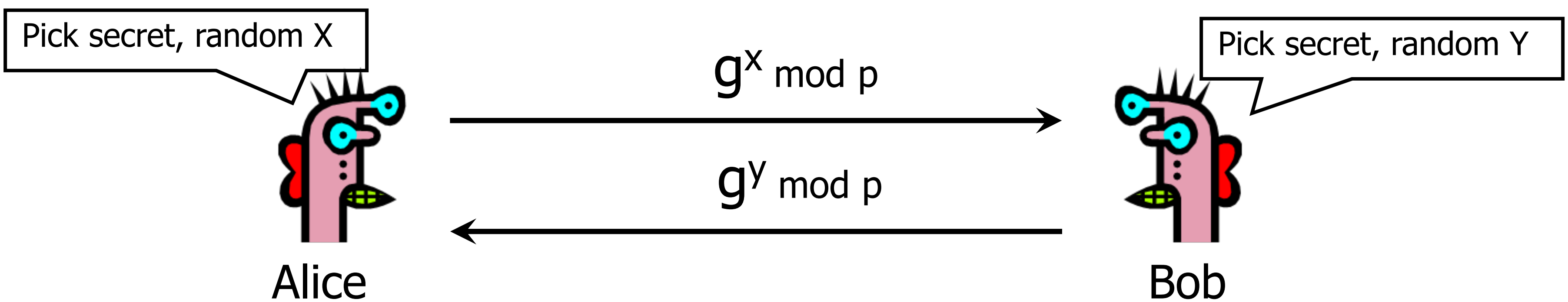
Diffie-Hellman Protocol (1976)

- Alice and Bob never met and share no secrets
- Public info: p and g
  - p is a large prime number, g is a generator of Z_p^*
    - $Z_p^*=\{1, 2 \ldots p-1\}$; $\forall a \in Z_p^*\ \exists i$ such that $a=g^i \bmod p$
    - Modular arithmetic: numbers "wrap around" after they reach p



Compute $k=(g^y)^x=g^{xy} \bmod p$

Compute $k=(g^x)^y=g^{xy} \bmod p$

Why Is Diffie-Hellman Secure?

- **Discrete Logarithm (DL)** problem:
  given g^x mod p, it's hard to extract x
  - There is no known efficient algorithm for doing this
  - This is not enough for Diffie-Hellman to be secure!
- **Computational Diffie-Hellman (CDH)** problem:
  given g^x and g^y, it's hard to compute g^{xy} mod p
  - … unless you know x or y, in which case it's easy
- **Decisional Diffie-Hellman (DDH)** problem:
  given g^x and g^y, it's hard to tell the difference between g^{xy} mod p and g^r mod p where r is random

DDH Assumption

- G is a group of large prime order q

  For $g_1,g_2,u_1,u_2 \in G$ define

$$DHP(g_1,g_2,u_1,u_2) = \begin{cases} 1 & \text{if } \exists x \in Z_q \text{ s.t. } u_1=g_1^x,\ u_2=g_2^x \\ 0 & \text{otherwise} \end{cases}$$

- Decisional Diffie-Hellman (DDH) Assumption says that there exists no efficient algorithm for computing DHP correctly with negligible error probability on all inputs

Security of Diffie-Hellman Protocol

- Assuming DDH problem is hard, Diffie-Hellman protocol is a secure key establishment protocol against <u>passive</u> attackers
  - Eavesdropper can't tell the difference between the established key and a random value
  - Can use new key for symmetric cryptography
    - Approx. 1000 times faster than modular exponentiation
- Basic Diffie-Hellman protocol does not provide authentication

Public-Key Encryption

- **Key generation:** computationally easy to generate a pair (public key PK, private key SK)
  - Computationally infeasible to determine private key SK given only public key PK
- **Encryption:** given plaintext M and public key PK, easy to compute ciphertext $C=E_{PK}(M)$
- **Decryption:** given ciphertext $C=E_{PK}(M)$ and private key SK, easy to compute plaintext M
  - Infeasible to compute M from C without SK
  - Trapdoor function: Decrypt(SK,Encrypt(PK,M))=M

When Is a Cipher "Secure"?

- Hard to recover the key?
  - What if attacker can learn plaintext without learning the key?
- Hard to recover plaintext from ciphertext?
  - What if attacker learns some bits or some function of bits?
- Fixed mapping from plaintexts to ciphertexts?
  - What if attacker sees two identical ciphertexts and infers that the corresponding plaintexts are identical?
  - Implication: encryption must be randomized or stateful

How Can a Cipher Be Attacked?

- Assume that the attacker knows the encryption algorithm and wants to decrypt some ciphertext
- Main question: what else does the attacker know?
  - Depends on the application in which cipher is used!
- Ciphertext-only attack
- Known-plaintext attack (stronger)
  - Knows some plaintext-ciphertext pairs
- Chosen-plaintext attack (even stronger)
  - Can obtain ciphertext for any plaintext of his choice
- Chosen-ciphertext attack (very strong)
  - Can decrypt any ciphertext <u>except</u> the target

The Chosen-Plaintext Game

- Attacker does not know the key
- He chooses as many plaintexts as he wants, and learns the corresponding ciphertexts
- When ready, he picks two plaintexts M_0 and M_1
  - He is even allowed to pick plaintexts for which he previously learned ciphertexts!
- He receives either a ciphertext of M_0, or a ciphertext of M_1
- He wins if he guesses correctly which one it is

CPA Game: Formalization

- Idea: attacker should not be able to learn even a single bit of the encrypted plaintext
- Define Enc(M_0,M_1,b) to be a function that returns encrypted M_b 0 or 1
  - Given two plaintexts, Enc returns a ciphertext of one or the other depending on the value of bit b
  - Think of Enc as a magic box that computes ciphertexts on attacker's demand. He can obtain a ciphertext of any plaintext M by submitting M_0=M_1=M, or he can try to learn even more by submitting $M_0 \neq M_1$.
- Attacker's goal is to learn just one bit b

Why Hide Everything?

- Leaking even a little bit of information about the plaintext can be disastrous
- Electronic voting
  - 2 candidates on the ballot (1 bit to encode the vote)
  - If ciphertext leaks the parity bit of the encrypted plaintext, eavesdropper learns the entire vote
- D-Day: Pas-de-Calais or Normandy?
  - Allies convinced Germans that invasion will take place at Pas-de-Calais
    - Dummy landing craft, feed information to double spies
  - Goal: hide a 1-bit secret

Chosen-Plaintext Security

- Consider two experiments (A is the attacker)

  Experiment 0

  A interacts with Enc(-,-,0) and outputs bit d

  Experiment 1

  A interacts with Enc(-,-,1) and outputs bit d

  - Identical except for the value of the secret bit
  - d is attacker's guess of the secret bit

- Attacker's advantage is defined as

  If A "knows" secret bit, he should be able to make his output depend on it

  | Prob(A outputs 1 in Exp0) - Prob(A outputs 1 in Exp1)) |

- Encryption scheme is chosen-plaintext secure if this advantage is negligible for any efficient A

Simple Example

- Any deterministic, stateless encryption scheme is insecure
  - Attacker can easily distinguish encryptions of different plaintexts from encryptions of identical plaintexts

Attacker A interacts with Enc(-,-,b)

Let X,Y be any two different plaintexts

$C_1 \leftarrow$ Enc(X,Y,b); $C_2 \leftarrow$ Enc(Y,Y,b);

If $C_1=C_2$ then b=1 else say b=0

- The advantage of this attacker A is 1

Prob(A outputs 1 if b=0)=0 Prob(A outputs 1 if b=1)=1

Semantic Security

[Goldwasser and Micali 1982]

- Hide all partial information
- Immune against attackers with a-priori knowledge about the plaintext
- Equivalent to ciphertext indistinguishability
  - It is infeasible to find two messages whose encryptions can be distinguished
  - Chosen-plaintext security is equivalent to ciphertext indistinguishability under the chosen-plaintext attack

Beyond Semantic Security

- Chosen-ciphertext security
  - "Lunch-time" attack [Naor and Yung 1990]
  - Adaptive chosen-ciphertext security [Rackoff and Simon 1991]
- Non-malleability [Dolev, Dwork, Naor 1991]
  - Infeasible to create a "related" ciphertext
  - Implies that an encrypted message cannot be modified without decrypting it

ElGamal Encryption

- **Key generation**
  - Pick a large prime p, generator g of Z^*_p
  - Private key: random x such that $1 \le x \le p-2$
  - Public key: (p, g, y), where $y = g^x \pmod p$
- **Encryption**
  - Pick random k, $1 \le k \le p-2$
  - $E(m) = (\gamma,\delta) = (g^k \bmod p,\ m \cdot y^k \bmod p)$
- **Decryption**
  - Given ciphertext (γ,δ), compute $\gamma^{-x} \bmod p$
  - Recover $m = \delta \cdot (\gamma^{-x}) \bmod p$

Semantic Security of ElGamal

- Semantic security of ElGamal encryption is equivalent to DDH
- Given an oracle for breaking DDH, show that we can find two messages whose ElGamal ciphertexts can be distinguished
- Given an oracle for distinguishing ElGamal ciphertexts, show that we can break DDH
  - Given a triplet $<g^a, g^b, Z>$, we can decide whether $Z=g^{ab} \bmod p$ or Z is random

DDH $\Rightarrow$ ElGamal

- Pick any two messages m_0, m_1
- Receive $E(m) = g^a, m \cdot y^a$
  - $y = g^x$ is the ElGamal public key
  - To break ElGamal, must determine if $m=m_0$ or $m=m_1$
- Run the DDH oracle on this triplet:

  $<g^a, y \cdot g^v, (m \cdot y^a) \cdot g^{av}/m_0> = <g^a, g^{x+v}, m \cdot g^{(x+v)a}/m_0>$
  - v is random
- If this is a DH triplet, then $m=m_0$, else $m=m_1$
- This breaks semantic security of ElGamal (why?)

ElGamal $\Rightarrow$ DDH (1)

- Suppose some algorithm A breaks ElGamal
  - Given any public key, A produces plaintexts m_0 and m_1 whose encryptions it can distinguish with advantage Adv
- We will use A to break DDH
  - Decide, given (g^a, g^b, Z), whether $Z=g^{ab}$ mod p or not
- Give $y=g^a$ mod p to A as the public key
- A produces m_0 and m_1
- We toss a coin for bit x and give A the ciphertext $(g^b, m_x \cdot Z)$ mod p
  - This is a valid ElGamal encryption of m_x **iff** $Z=g^{ab}$ mod p

ElGamal $\Rightarrow$ DDH (2)

- A receives $(g^b, m_x \cdot Z)$ mod p
  - This is a valid ElGamal encryption of m_x **iff** $Z=g^{ab}$ mod p
- A outputs his guess of bit x (why?)
- If A guessed x correctly, we say that $Z=g^{ab}$ mod p, otherwise we say that Z is random
- What is our advantage in breaking DDH?
  - If $Z=g^{ab}$ mod p, we are correct with prob. Adv(A)
  - If Z is random, we are correct with prob. ½
  - Our advantage in breaking DDH is Adv(A)/2