Big-oh stuff

Definition You should know this definition *by heart* and be able to give it, if asked.

> A function $f(n)$ is $O(g(n))$ (pronounced "big-oh") of a function $g(n)$ if there exists positive constants C_1 and C_2 such that $f(n) \leq C_1 g(n)$ whenever $n > C_2$.

This is the *definition* of what it means to say $f(n)$ is $O(g(n))$. So, a real proof that $f(n)$ is $O(g(n))$ requires providing the constants C_1 and C_2 and proving the result above.

Most of the time you won't be asked to provide the constants, but rather to be able to guess intelligently (and back up your guess if asked) whether a function is big-oh of another function.

But now that you know the definition of what big-oh means, you can see the following statements will be true. Here we will assume $f(n)$ is $O(g(n))$, and that C_1 and C_2 are defined so that $f(n) \leq C_1 g(n)$ whenever $n > C_2$.

1. $$lim_{n\to\infty} \frac{f(n)}{g(n)} \leq C_1$$

2. For $n > C_2$, $\log f(n) < \log(C_1) + \log g(n)$. Therefore,

   $$lim_{n\to\infty} \log f(n) - \log g(n) \leq \log(C_1).$$

   (Note that this analysis does not depend upon the base for the log; this means that you can use ln (the natural log, which is $\log_e$), or log base 2 (written $\log_2$), or any base you like.)

These two results may help you find the constant C_1. However, what they suggest is that you should be able to compute limits, something you may not right now be comfortable with. Furthermore, just knowing that the limit exists (which it may not) doesn't make it straightforward to pick the constant.

We start with the "easy" case, where the limit exists. Suppose you have the first case, where you are computing $lim_{n\to\infty} \frac{f(n)}{g(n)}$, and you find it is at most C (for some constant C. Should you pick C to be C_1? **NO**, you should not. And why?

Consider for example the following pair of functions

- $f(n) = n^2 + 5$
- $g(n) = n^2$.

If you compute $\lim_{n\to\infty} \frac{f(n)}{g(n)}$, you will get 1. But setting $C_1 = 1$ won't work. The reason is that that ratio approaches its limit *from above*. So you need to pick some constant greater than the limit, not equal to the limit. (Conversely, if the ratio approaches its limit from below, you can pick the constant C_1 to be that limit, but setting it to be bigger is always safe.)

Here's another example.

- $f(n) = 3^{\sqrt{n}}$
- $g(n) = 2^n$

Trying to figure out whether $f(n)$ is $O(g(n))$ using $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ gives you something harder to compute. You'll need to use L'Hopital's rule, but that may be something you are not comfortable with in this context. Here, let me suggest you use the second approach – taking logs. (Even this you may not be comfortable with!)

Let's take logs using base 2. Then we get

- $\log_2(3^{\sqrt{n}}) = \sqrt{n}\log_2(3)$. Note that $\log_2(3) < 2$, so this is less than $2\sqrt{n}$.
- $\log_2(2^n) = n$

We continue:

$$\log_2(f(n)) - \log_2(g(n)) =$$

$$\sqrt{n}\log_2(3) - n <$$

$$2\sqrt{n} - n$$

Note that when $n > 4$, $\sqrt{n} > 2$, and so $2\sqrt{n} < n$. Hence,

$$2\sqrt{n} - n < 0 \textit{ when } n > 4.$$

How do we use this? The analysis given above shows that if $f(n)$ is $O(g(n))$, then $\log f(n) - \log g(n) < \log C_1$ for large enough n. So we just have to pick C_1 so that $\log C_1 \geq 0$. *What values of C_1 satisfy this? Answer: all $C_1 \geq 1$.* Setting $C_1 = 1$ then makes sense. What value would you give for C_2? The analysis here shows that $C_2 = 4$ works.

Skills you will need The skills you need are mostly from pre-calculus and calculus, and you are probably rusty. Please practice!

- You will need to be able to compute logarithms using any base.
- You will need to use L'Hopitals Rule.
- You need to be able to compute limits.

Preparing for the quiz Try to answer each of the following questions, any of which could appear on the quiz. You could expect questions like these, even if they are not identical.

1. Provide the definition of the set of functions $f(n)$ that are $O(n^2)$.
2. Provide the constants proving that 3^n is $O(3^n - 2^n)$.
3. Solve for $H(n)$: $3^{n^2-1} = 4^{H(n)}$
4. Compute $\lim_{n\to\infty} \frac{\ln n}{n}$
5. Compute $\log_2(3f(n)^n)$
6. Compute $\log_3(5n^2 4^n)$
7. Determine (no proof requested), for each pair of functions below, whether (a) $f(n)$ is $O(g(n))$ but not vice-versa, (b) $g(n)$ is $O(f(n))$ but not vice-versa, (c) both are big-oh of each other, or (d) neither is big-oh of each other You should only concern yourself with values $n \geq 1$.
   - $f(n) = n^2$ and $g(n) = \log(n^n)$
   - $f(n) = (\log n)^n$ and $g(n) = \sqrt{n}$
   - $f(n) = \log n^{500}$ and $g(n) = 100$
   - $f(n) = 100 + 3/n$ and $g(n) = 5$
8. Sally says $f(n)$ is $O(g(n))$, but Bob notes that $f(n) > g(n)$ for all n, and so says $f(n)$ cannot be $O(g(n))$. What do you think of this argument? Assuming that $f(n) > g(n)$ is true, can Sally be right? Or is Bob right?