Induction, etc.

January 29, 2009

Prof. Tandy Warnow

Topics for today:

- The rock game
- Proofs by induction
- Recurrence relations

The Rock Game

- Two person game
- Starting condition: two piles of rocks
- Each person can do one of the following: remove one rock from each pile, or remove one rock from exactly one pile
- The person to remove the last rock wins

Who wins the Rock Game?

- Assume each player plays optimally.
- The first player wins if (and only if) it’s always possible for the first player to win, no matter what the second player does.
- The second player wins if (and only if) it’s always possible for the second player to win, no matter what the first player does.

The Rock Game

- Yes/No problem: Given the number of rocks in each pile, determine if the first player wins
- Construction problem: For the same input, devise a winning strategy for the player who wins.

Dynamic programming

- Fill in a matrix RockWinner(i,j) with entries "1" or "2" indicating whether the first player or the second player wins.
- Base cases:
  - RockWinner(1,1) =1
  - RockWinner(1,0)=RockWinner(0,1)=1
- How do you figure out RockWinner(i,j) from "smaller" subproblems?

Induction

- Induction is a basic proof technique, which you can use to:
  - Design algorithms and prove them correct
  - Analyze running times of algorithms defined recursively

Induction proofs

- I want a closed form solution to 1+2+3+…+n.
- How do I find this?
- How do I prove my closed form solution is correct?

Induction, cont.

- Someone tells me $1+2+\ldots+n=n(n+1)/2$

Induction, cont.

- Someone tells me 1+2+…+n=n(n+1)/2
- I check the first few cases:
  - n 1+2+…+n n(n+1)/2
  - 1 1 1x2/2=1
  - 2 3 2x3/2=3
  - 3 6 3x4/2=6

Induction, cont.

- Someone tells me 1+2+…+n=n(n+1)/2
- I check the first few cases:

n	1+2+…+n	n(n+1)/2
1	1	1x2/2=1
2	3	2x3/2=3
3	6	3x4/2=6

- I am feeling somewhat more convinced. But -- how do I convince myself the formula is correct?

Induction, cont.

- Induction is a proof technique, which (in a simple form) amounts to the following:
- Suppose I want to confirm some statement P(n) which depends upon the parameter n, where n is a natural number (1,2,3,…).
- The proof technique operates in two steps:
  - I verify that P(1) is true
  - I prove that no matter what the value is for N, if P(N) is true, then P(N+1) is true.
- From this we can deduce that P(n) is ***always true (i.e. there is no n for which P(n) is false)***. Note: the assertion that P(N) is true is called the "Inductive Hypothesis" (IH).

Why induction works

If P(n) is sometimes false, then there has to be a smallest n (which we'll call **X**) for which it is false. But then either **X**=1 (the smallest value we consider) or **X**>1.

- Since we check the *base case* (n=1) we know that P(1) is true. So **X**>1.
- Since X is the smallest value for n for which P(**X**) is false, we know P(**X**-1) is true.
- The *induction step* shows that if P(**X**-1) is true, then P(**X**) is true -- contradicting the assumption that P(n) is sometimes false.

Back to proving $1+2+3+\ldots+n=n(n+1)/2$

- We check the base case, n=1, and it works.
- Now assume the statement is true for a particular value X, and see if we can prove that it must be true for X+1.
- Statement for X assumed true
  - $1+2+3+\ldots+X=X(X+1)/2$.
- We try to **prove** statement for X+1:
  - $1+2+3+\ldots+(X+1)=(X+1)(X+2)/2$

Proof, cont.

$1+2+\ldots+(X+1)$

$= 1+2+\ldots+X+(X+1)$ (by definition)

$= X(X+1)/2 + (X+1)$ (by the Ind. Hyp.)

$= (X+1)(X/2 + 1)$ (arithmetic)

$= (X+1)(X+2)/2$ (arithmetic)

Which is the statement P(X+1) (q.e.d.)!

Another proof by induction

- Consider the sequence defined by
  - $T(1)=1$, $T(2)=3$,
  - $T(n)=T(n-2)$, $n>2$
  - The sequence is 1,3,1,3,1,3,1,3,…
- What is the solution (closed form) for $T(n)$?

Another proof by induction

The sequence defined by

- $T(1)=1$, $T(2)=3$,
- $T(n)=T(n-2)$, $n>2$
- i.e., sequence $T(1)=1$, $T(2)=3$, $T(3)=1$, $T(4)=3$, $T(5)=1$, $T(6)=3$, $T(7)=1,\ldots$

has hypothesized closed form:

$T(n)=1$ for n odd, $T(n)=3$ for n even

- We want to prove the solution is correct!
- Note: we need a “stronger” form of induction, where we establish a set of base cases, and we assume that the statement P(n) is true for *all* smaller values.

Proof by strong induction

- We will prove that
  - T(n)=1 if n is odd, and
  - T(n)=3 if n is even
- Proof technique:
- We establish the base cases (for n=1 and 2).
- We assume that the statement holds for all n=1,2,…, X, and we show it's true for n=X+1.

Proof, cont.

- Base cases: n=1 and 2 are easy.
- Now assume that for some X (which is at least 2), and for all n=1,2,…,X
  - T(n)=1 if n odd, and
  - T(n)=3 if n even.
- So X+1 is at least 3.
- T(X+1)=T(X-1) by definition. Hence, T(X+1)=1 for X-1 odd, and T(X+1)=3 for X-1 even.
- Note that X+1 and X-1 have the same parity.
- Hence, T(X+1)=1 if X+1 is odd, and T(X+1)=3 if X+1 is even.
- Q.e.d.

Recurrence relations

- Functions can be defined recursively.
- Examples
  - $f(n)=3f(n-1)$ if $n>1$, $f(1)=2$
  - $g(n)=g(n-1)+3$ if $n>1$, $g(1)=4$
  - $h(n)=3h(n/2)$ if $n>1$, $h(1)=2$

    (where $n/2$ is really the floor of $n/2$)
  - $k(n)=4k(n-1)+n$ if $n>1$, $k(1)=5$
  - $m(n)=m(n-1)+m(n-2)$ if $n>2$, $m(1)=m(2)=1$
  - $p(n)=p(n-1)+2n$ if $n>1$, $p(1)=1$
- We would like closed form solutions (or closed form bounds) for these functions.

Solving recursions

- $f(n)=3f(n-1)$ for $n>1$, $f(1)=2$
- Sequence: $f(1)=2$, $f(2)=6$, $f(3)=18$, $f(4)=54$, $f(5)=162$,…
- Hypothesized closed form: $f(n)=2\text{x}3^{n-1}$
- Prove this by induction!

Another recurrence

- $p(n)=p(n-1)+2n$ for $n>1$, $p(1)=1$
- Sequence: $p(1)=1$, $p(2)=5$, $p(3)=11$, …

Hard to see the closed form for the function!

Try expanding!

Solving the recurrence

- $p(n) = p(n-1)+2n$ for $n>1$, $p(1)=1$.
- Therefore, $p(n-1) = p(n-2)+2(n-1)$
- Hence,

  $p(n) = (p(n-2)+2(n-1)) + 2n$

  $= p(n-2)+4n-2$
- Expand $p(n)$ in terms of $p(n-i)$ for $i=1,2,3, 4$ (and larger if you have patience) to look for a pattern.

Solving the recurrence

$p(n) = p(n-1) + 2n$

$= p(n-2)+ 4n-2$

$= p(n-3) + 6n-6$

$= p(n-4) + 8n-12$

- Looks quadratic: $p(n)=q(n)=an^2+bn+c$.
- Solve for a,b,c, to produce the hypothesized closed form quadratic $q(n)$, and then try to prove $p(n)=q(n)$ for all n.

Solving for a,b,c

- p(n) = p(n-1)+2n, p(1) = 1, so p(2) = 5 and p(3) = 11.
- Solving for a,b,c we get

  a+b+c=1

  4a+2b+c=5

  9a+3b+c=11

  Solving, we find a=1, b=1, c=-1, i.e., we obtain the polynomial $q(n)=n^2+n-1$.
- Now we check if p(n)=q(n) for all n. How?

Proof by induction

- We prove $p(n)=q(n)$ for all n, where $q(n)=n^2+n-1$, and $p(n)$ is defined by $p(n)=p(n-1)+2n$ for $n>1$, and $p(1)=1$.
- Base case: $n=1$. Inspection.
- Induction step: assume $p(n)=q(n)$ and see if we can deduce that $p(n+1)=q(n+1)$.

Proof, cont.

$p(n+1) = p(n)+2(n+1)$	by definition
$= q(n) + 2(n+1)$	by I.H.
$= [n^2+n-1] + 2(n+1)$	by definition
$= n^2+3n+1$	arithmetic
$= [n^2+2n+1] + [n+1]-1$	arithmetic
$= (n+1)^2+(n+1)-1$	arithmetic
$= q(n+1)$	definition

Another recursion

- $h(n)=3h(n/2)$ for $n>1$, $h(1)=2$, where $n/2$ is the floor of $n/2$
- Sequence: $h(1)=2$, $h(2)=6$, $h(3)=6$, $h(4)=18$, $h(5)=18$, $h(6)=18$, $h(7)=18$, $h(8)=54$, $h(9)=54$, …
- Hypothesized closed form solution:
  - $h(n)=2\text{x}3^{g(n)}$, where $g(n)=\text{floor}(\log_2 n)$

Another recursion

- $k(n) = 4k(n-1)+n$ for $n>1$, $k(1)=5$
- Sequence: $k(1)=5$, $k(2)=22$, $k(3)=91$, $k(4)=368$, …
- Closed form solution?
- Bounds (lower or upper) on $k(n)$?

Bounds

- $k(n) = 4k(n-1)+n$ for $n>1$, $k(1)=5$
- We conjecture $k(n) \geq 4^{n-1}$ for all n.
- Base case: $n=1$ (inspection)
- Inductive hypothesis: $k(N) \geq 4^{N-1}$
- We try to prove $k(N+1) \geq 4^{N}$

Bounds, cont.

Inductive hypothesis: $k(N) \geq 4^{N-1}$

We try to prove $k(N+1) \geq 4^{N}$

$k(N+1)=4k(N)+N$	by definition
$\geq 4^{N}+N$	by I.H.
$> 4^{N}$	arithmetic

Summary

- Induction is a proof technique
- You can use it to prove equations or inequalities
- (Not yet shown: you can use it to prove algorithms correct, even when those algorithms aren't obviously based upon natural numbers!)
- However, figuring out what you want to prove requires work (and sometimes guessing).