Disjunctive Defaults

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Abstract

A generalization of Reiter's default logic is proposed that provides an improved treatment of default reasoning with disjunctive information. The new system — *the disjunctive default logic* — is used in the paper to reexamine the "broken-hand" example of Poole. We also compare the expressive power of this approach with two other approaches which interpret disjunctive information within the standard default logic. Finally, we show that our semantics of disjunctive default logic is a generalization of the semantics of disjunctive and extended disjunctive databases.

1 INTRODUCTION

In this paper we generalize the theory of default reasoning developed by Reiter [Rei80]. The generalization is motivated by a difficulty encountered in attempts to use defaults in the presence of disjunctive information [Poo89]. The difficulty has to do with the difference between a default theory with two extensions — one containing a sentence α, the other a sentence β — and the theory with a single extension, containing the disjunction $\alpha \vee \beta$. This difficulty was also observed by Lin and Shoham in [LS90]. They present there an example (Example 3.1 [LS90]) of a (modal) theory T, containing disjunctive information, and comment that no default theory exists that corresponds to T.

This difference may seem inessential, because the set of *theorems* (sentences that belong to all extensions) may be the same in both cases. But a default theory is more than just a set of theorems. The usual explanation of the intuitive meaning of a default mentions *consistency* — consistency with a set of beliefs. Defaults are assertions not only about a domain described by the underlying first-order language, but also about a set of beliefs concerning that domain. A default theory with several extensions is a theory of several sets of beliefs. To formalize a body of knowledge in default logic, one has to decide what these sets of beliefs are. In particular, one needs to know whether the theory is about two sets of beliefs, or about one set containing disjunctive information. Poole's example, reproduced in Section 3 below, illustrates the crucial role of this distinction.

In the paper, we introduce an extension of Reiter's default logic by adding capabilities of handling disjunctive information. This new system — *disjunctive default logic* — is formally introduced in Section 5. We also show there how to use this new system to formalize Poole's example. Next, in Section 6, we study the expressive power of our formalization by comparing it with two systems obtained by interpreting a disjunctive default theory in the standard default logic. Finally, we show that our semantics of disjunctive default logic is a generalization of the semantics of disjunctive databases and extended disjunctive databases proposed in [GL90a]. Disjunctive default logic can be extended even further to the language of modal logic. This approach makes proofs of some of our results simpler and more elegant. We introduce this modal extension of disjunctive default logic in Section 8. Proofs of our results are gathered there, too.

2 DEFAULT THEORIES

We begin with a brief review of Reiter's default logic restricted, for simplicity, to the case of quantifier-free defaults. This restriction allows us to disregard the process of Skolemization, involved in defining extensions in the general case; see [Rei80], Section 7.1. A *default* is an expression of the form

$$\frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma}, \tag{1}$$

where $\alpha, \beta_1, \ldots, \beta_m, \gamma$ ($m \geq 0$) are quantifier-free formulas. Formula α is the *prerequisite* of the default, $\beta_1, \ldots, \beta_m$ are its *justifications*, and γ is its *consequent*. Informally, (1) is interpreted as follows:

If α holds, and
each of $\beta_1, \ldots, \beta_m$
can consistently be assumed,
then infer γ. (2)

If the prerequisite α in (1) is the formula *true*, it will be dropped; if, in addition, $m = 0$, then we identify the default (1) with its consequent γ. Thus formulas can be viewed as defaults of a special form. A *default theory* is a set of defaults. Let us comment here that our definition differs from the standard definition of a default theory as a pair (D, W), where D is a set of defaults and W is a collection of (quantifier-free) formulas. As stated above, formulas can be viewed as defaults of a special form, and both approaches can easily be shown to be equivalent.

Reiter's definition of an extension shows how to make (2) precise. It can be stated as follows.

Definition 2.1 *Let D be a default theory, and let E be a set of sentences. E is an extension for D if it coincides with the smallest deductively closed set of sentences E' satisfying the condition: for any ground instance (1) of any default from D, if $\alpha \in E'$ and $\neg\beta_1, \ldots, \neg\beta_m \notin E$ then $\gamma \in E'$. A theorem of a default theory is a sentence that belongs to each of its extensions.*

Notice that the definition of an extension treats a default with variables as shorthand for the set of its ground instances[1].

It is often convenient to think about this definition of extensions in terms of the following two-step procedure. Let D be a default theory. The first step, which turns out to be responsible for nonmonotonicity, is to assume a hypothetical belief set E and to preprocess the set of defaults in D with respect to E.

Definition 2.2 *Let D be a default theory and let E be a set of sentences. The* reduct of D with respect to *E, denoted D^E, is the set of inference rules defined as follows: An inference rule*

$$\frac{\alpha}{\gamma}$$

is in D^E if for some $\beta_1, \ldots, \beta_m$ such that for every i, $1 \leq i \leq m$, $\neg\beta_i \notin E$, the default

$$\frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma}$$

is in D.

[1] A different approach to default logic, treating variables in defaults as "real" variables and not only as symbols for objects explicitly appearing in the theory was proposed in [Lif90]

The second step is to define the set of theorems of the formal system obtained by expanding the system of propositional logic by the rules in D^E. This set of theorems can be defined as the set of all formulas having proofs in this system or, equivalently, as the smallest theory E' closed under provability in propositional calculus and under the rules from D^E, where E' is *closed under the rule*

$$\frac{\alpha}{\gamma}$$

if whenever $\alpha \in E'$, $\gamma \in E'$, as well. If this set of theorems coincides with E, E is an extension, and conversely. Formally, we have the following theorem

Theorem 2.3 *A set of sentences E is an extension for a default theory D if and only if E is the minimal set E' closed under provability in propositional calculus and under the rules from D^E.*

3 POOLE'S EXAMPLE

The following example from [Poo89] illustrates a difficulty that arises in some attempts to use Reiter's formalism in the presence of disjunctive information.

By default, people's left arms are usable, but a person with a broken left arm is an exception, and similarly for the right arms. One way to express this in Reiter's notation is to use "semi-normal" defaults, in the spirit of [RC81]. Below, *lh* and *rh* stand for left hand and right hand, respectively.

$$\frac{: \textit{lh-usable} \wedge \neg \textit{lh-broken}}{\textit{lh-usable}}, \quad \frac{: \textit{rh-usable} \wedge \neg \textit{rh-broken}}{\textit{rh-usable}}. \tag{3}$$

If you have no additional information about my arms, you will conclude that they are usable. Indeed, the default theory (3) has a single extension, containing *lh-usable* and *rh-usable*. If you know that my left arm is broken,

$$\textit{lh-broken} \tag{4}$$

then this conclusion is blocked — the extension for the default theory (3), (4) contains *rh-usable*, but not *lh-usable*. So far Reiter's logic works fine.

But suppose you remember seeing me with a broken arm, you are not sure which one:

$$\textit{lh-broken} \vee \textit{rh-broken}. \tag{5}$$

The default theory (3), (5) has a single extension also. In addition to (5), this extension contains, unfortunately, both *lh-usable* and *rh-usable*, contrary to what we would expect.

There are other natural formalizations of Poole's example in standard default logic. One such formalization, using new variables ab_1 and ab_2 to denote abnormality, consists of the formula (5), two formulas (6) and two defaults (7).

$$\textit{lh-broken} \Rightarrow ab_1, \qquad \textit{rh-broken} \Rightarrow ab_2 \tag{6}$$

$$\frac{:\neg ab_1}{\textit{lh-usable}}, \qquad \frac{:\neg ab_2}{\textit{rh-usable}}. \tag{7}$$

Also this default theory has a unique extension containing both *lh-usable* and *rh-usable*, which does not agree with the intuition.

4 DISCUSSION

Let us apply the informal interpretation (2) to Poole's example. The first of the defaults (3) says: If it is consistent to assume $\textit{lh-usable} \wedge \neg \textit{lh-broken}$, then infer *lh-usable*. This mode of reasoning is acceptable when all available information about *lh-broken* is "vivid" — expressed by one of the literals *lh-broken*, $\neg$*lh-broken* (see [Lev86]). The disjunction (5) suggests *lh-broken* only as a possibility. This does not make the opposite assumption inconsistent, although we do not want the default to be applicable when this disjunction is postulated. Consistency in the defaults (2) is meant to be consistency with a set of beliefs which is not expected to contain disjunctive information. This explains why the defaults (3) lead, in the presence of (5), to unintended conclusions.

The way out is to express the incompleteness of information by the multiplicity of extensions, rather than by a single extension containing a disjunction. What we want in this example is two extensions: one containing *lh-broken*, the other *rh-broken*. Such a formalization would be similar to the default theory (3), (5), in that the disjunction (5) would be among its theorems. The difference is that each extension would be the deductive closure of a set of literals.

In the new formalism, the postulate (5) will be replaced by the expression

$$\textit{lh-broken} | \textit{rh-broken}. \tag{8}$$

Semantically, the difference between (5) and (8) is that the latter requires an extension to contain one of the two disjunctive terms, rather than the disjunction. This is similar to the difference between the assertions: "$\alpha \vee \beta$ is known" and "α is known or β is known."

5 DISJUNCTIVE DEFAULT THEORIES

In this section we introduce the main concepts of the paper — a disjunctive default, a disjunctive default theory. Most importantly, we extend Reiter's definition of extension to the case of disjunctive defaults. Next, we use our approach to analyze Poole's example.

A *disjunctive default* is an expression of the form

$$\frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma_1 | \ldots | \gamma_n}, \tag{9}$$

where $\alpha, \beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n$ $(m, n \geq 0)$ are quantifier-free formulas. Formula α is the *prerequisite* of the default, $\beta_1, \ldots, \beta_m$ are its *justifications*, and $\gamma_1, \ldots, \gamma_n$ are its *consequents*. If the prerequisite α in (1) is the formula *true*, it will be dropped; if, in addition, $m = 0$, then we write the default (1) as $\gamma_1 | \ldots | \gamma_n$. A *disjunctive default theory* (a *ddt*, for short) is a set of disjunctive defaults.

Definition 5.1 *Let D be a disjunctive default theory, and let E be a set of sentences. E is an extension for D if it is one of the minimal deductively closed sets of sentences E' satisfying the condition: For any ground instance (9) of any default from D, if $\alpha \in E'$ and $\neg\beta_1, \ldots, \neg\beta_m \notin E$ then, for some i (1øiøn), $\gamma_i \in E'$. A theorem is a sentence that belongs to all extensions.*

Observe that in the definition of an extension, disjunctive defaults are treated as shorthand for the sets of their ground instances.

It is clear that for standard (nondisjunctive) default theories this definition gives extensions in the usual sense, as defined in Section 2.

The definition of an extension for a *ddt* can also be described by means of the concept of *reduct*. To this end, we need more terminology. A *disjunctive rule* is an expression of the form

$$\frac{\alpha}{\gamma_1 | \ldots | \gamma_n}. \tag{10}$$

We say that a theory E is *closed under a disjunctive rule* (10) if, whenever $\alpha \in E$, then there exist i, $1 \leq i \leq n$, $\gamma_i \in E$.

Remark Note that given a set S of disjunctive rules there may be several minimal sets of sentences closed under S, unlike in the case of sets of standard inference rules where there always exists the least set of sentences closed under them.

Definition 5.2 *Let D be a ddt and let E be a set of sentences. The reduct of D with respect to E, denoted D^E, is the set of inference rules defined as follows: An inference rule*

$$\frac{\alpha}{\gamma_1 | \ldots | \gamma_n}$$

is in D^E if for some $\beta_1, \ldots, \beta_m$ such that $\neg\beta_i \notin E$, $1 \leq i \leq m$, the default

$$\frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma_1 | \ldots | \gamma_n}$$

is in D.

We have the following straightforward theorem.

Theorem 5.3 *A set of sentences E is an extension for a ddt D if and only if E is a minimal set closed under propositional consequence and under the rules from D^E.*

The difference between disjunctive defaults and standard default with disjunctive consequents can be illustrated by the following example from [Rei80]. The default theory

$$\{\frac{a : b}{b},\ \frac{c : d}{d},\ a \vee c\},$$

where a, b, c, d are propositional symbols, has a single extension, consisting of the disjunction $a \vee c$ and its logical consequences; the first two defaults "don't work." On the contrary, the disjunctive default theory

$$\{\frac{a : b}{b},\ \frac{c : d}{d},\ a|c\} \tag{11}$$

(where $a|b$ is to be understood as a disjunctive default according to the convention introduced at the beginning of the section) has two extensions: the deductive closure of $\{a, b\}$ and the deductive closure of $\{c, d\}$. The formula $b \vee d$ belongs to both extensions, and consequently is a theorem.

The new formalization of Poole's example is the disjunctive default theory (3), (8). It has two extensions. Each of the extensions contains one of the atoms *lh-usable*, *rh-usable* and does not contain the other. Thus neither atom is a theorem, and Poole's paradox is eliminated.

It is also easy to see that disjunctive default logic is exactly what is needed to properly represent the theory of Example 3.1 of [LS90].

6 EXPRESSING DISJUNCTIVE INFORMATION BY STANDARD DEFAULTS

As we mentioned earlier, defaults are assertions not only about a domain described by the underlying first-order language, but also about possible sets of beliefs concerning that domain. In other words, we are interested not only in the theorems of a *ddt* D, but also in what sets of beliefs can be associated with D. The class of extensions for a *ddt* D, as defined in the previous section, can be viewed as the sets of beliefs "grounded" in the default theory D.

An important question is: Can the behavior of disjunctive defaults be simulated in standard default logic? More precisely, is there a general transformation that assigns to a *ddt* D a standard default theory D' (or a family $\mathcal{D}$ of standard default theories) so that extensions for D are exactly extensions for D' (or for theories in $\mathcal{D}$).

We will consider two naturally arising transformations. The first of them "breaks" a disjunctive default

$$\frac{\alpha :\ \beta_1, \ldots, \beta_n}{\gamma_1 | \ldots | \gamma_n}$$

into n standard defaults

$$\frac{\alpha : \beta_1, \ldots, \beta_m, \neg\gamma_2, \ldots, \neg\gamma_n}{\gamma_1},$$

$$\ldots$$

$$\frac{\alpha : \beta_1, \ldots, \beta_m, \neg\gamma_1, \ldots, \neg\gamma_{n-1}}{\gamma_n}.$$

Given a *ddt* D, by D' we denote the default theory obtained by applying this transformation to every default in D. For example, for

$$D = \{\frac{: b}{a|b}, \frac{: a}{a|b}\}$$

we have

$$D' = \{\frac{: b, \neg b}{a}, \frac{: b, \neg a}{b}, \frac{: a, \neg b}{a}, \frac{: a, \neg a}{b}\}.$$

It is easy to see that D and D' have the same extensions. If by $Cn(\cdot)$ we denote the operator of propositional consequence, then these extensions are $Cn(\{a\})$ and $Cn(\{b\})$.

But in general, this is not the case. Consider the following theory:

$$D = \{a \Leftrightarrow b,\ a \mid b\}.$$

Then,

$$D' = \{a \Leftrightarrow b,\ \frac{: \neg a}{b},\ \frac{: \neg b}{a}\}.$$

The theory $Cn(\{a, b\})$ is the unique extension for D but D' has no extensions. Thus, the classes of extensions for D and D' are not, in general, identical. However, the following weaker property holds.

Theorem 6.1 *If E is an extension for D' then E is an extension for D.*

Remark: This theorem implies that an interpretation of a *ddt* D which associates with it the class of extensions for D' has "more" theorems than the logic of disjunctive defaults with extensions as defined in Section 5.

There is yet another natural way of looking at a disjunctive default theory in the standard default logic. A *ddt* D can be viewed as a "disjunction" of a collection of standard default theories. Namely, the family $\mathcal{D}$ of all standard default theories that can be obtained from D by dropping, in each of its defaults, all but one of the consequents. Every such standard default theory is called a *cover* of D. Take now all extensions for all covers of D. Will we get the set of all extensions for D?

In many cases, yes. For instance, this procedure will replace the *ddt* given by (11) by two standard default theories:

$$\{\frac{a:b}{b},\ \frac{c:d}{d},\ a\}$$

and

$$\{\frac{a:b}{b},\ \frac{c:d}{d},\ c\}.$$

Theory $Cn(\{a,b\})$ is the unique extension for the first theory and $Cn(\{c,d\})$ is the unique extension for the second one. This collection coincides with the collection of extensions for the theory given by (11).

But in general, this equality does not hold. Consider the disjunctive default theory

$$\{a \mid b,\ \frac{a:}{b},\ \frac{:\neg a}{c}\}. \tag{12}$$

The theory $Cn(\{b,c\})$ is an extension for one of the corresponding covers:

$$\{b,\ \frac{a:}{b},\ \frac{:\neg a}{c}\}.$$

In addition, no smaller set of sentences closed under propositional consequence is an extension for any cover of the theory (12). Yet, theory $Cn(\{b,c\})$ is not an extension for (12). Nevertheless, we have the following weaker result.

Theorem 6.2 *If E is an extension for D, then E is a minimal (with respect to inclusion) element in the class of all extensions for the covers of D.*

Remark Thus, if we associate with a *ddt* D the collection of all minimal (with respect to inclusion) elements in the class of all extensions for the covers of D, the resulting system is "more secure" (has less theorems) than the system based on the class of extensions for D, as defined in Section 5.

There is however an important class of disjunctive default theories for which both systems are equivalent.

Theorem 6.3 *If a ddt D consists only of justification-free defaults, then the classes of extensions for D and of minimal (with respect to inclusion) extensions for covers of D coincide.*

7 RELATION TO DISJUNCTIVE DATABASES

Disjunctive default logic with extensions as defined in Section 5 can be viewed as a generalization of the semantics for disjunctive databases proposed in [GL90a]. An *extended disjunctive database* is a set of rules of the form

$$c_1 \mid \ldots \mid c_n \leftarrow a_1, \ldots, a_k, not\ b_1, \ldots, not\ b_m, \tag{13}$$

where $n, m, k \geq 0$, and a_i, b_i and c_i are literals. (The word "extended" points to the fact that the literals can be negative.) The semantics of such databases is a generalization of the "answer set" semantics for logic programs with classical negation defined in [GL90b].

Definition 7.1 *Let P be an extended disjunctive database, and let M be a set of literals Set M is an answer set for P if it is one of the minimal sets of literals M' satisfying the conditions:*

1. *for any rule (13) in P, if $a_i \in M'$, $1 \leq i \leq k$, and $b_i \notin M$, $1 \leq i \leq m$, then, for some i, $1 \leq i \leq n$, $c_i \in M'$;*
2. *if for some atom a,*

   $$a, \neg a \in M',$$

   then M' contains all literals.

An extended disjunctive database P can be associated with a *ddt* $emb(P)$ obtained from P by replacing each rule r with its disjunctive default interpretation $emb(r)$, where for a rule r given by (13) we have

$$emb(r) = \frac{a_1 \wedge \ldots \wedge a_k : \neg b_1, \ldots, \neg b_m}{c_1 \mid \ldots \mid c_n}.$$

We have the following theorem. It generalizes similar theorems on embedding logic programs and logic programs with classical negation in default logic [BF88, GL90b].

Theorem 7.2 *Let P be an extended disjunctive database. A set of literals M is an answer set of P if and only if M is the set of all literals from an extension for $emb(P)$.*

8 PROOFS

Our definition of an extension for a *ddt*, as well as Reiter's definition of an extension for a (standard) default theory, treat defaults as shorthand for the set of their ground instances. Ground instances of defaults are variable-free and, thus, without loss of generality we will restrict in the proofs to the case when defaults are built of formulas of fixed propositional language $\mathcal{L}$.

Proofs of Theorems 2.3 and 5.3 are straightforward and are omitted.

Theorem 6.1 *If E is an extension for D' then E is an extension for D.*

Proof: Clearly, E is an extension for D'^E. In particular, E is closed under propositional consequence operator. In addition, E is closed under all rules in D'^E. Consider now a rule

$$\frac{\alpha}{\gamma_1|\cdots|\gamma_n} \tag{14}$$

from D^E. If E contains at least one γ_i, then E is closed under the rule (14). If E does not contain any γ_i, then rules

$$\frac{\alpha}{\gamma_1}, \ldots, \frac{\alpha}{\gamma_n}$$

are all in D'^E. Since E is closed under the rules from D'^E, it follows that $\alpha \notin E$. Thus, E is closed under all rules in D^E.

Consider an arbitrary theory $E' \subseteq E$ closed under propositional consequence and the rules in D^E. Consider a rule

$$\frac{\alpha}{\gamma}$$

from D'^E. Then, there is a default

$$\frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma | \gamma_2 | \ldots | \gamma_n}$$

in D, such that $\neg\beta_i \notin E$, $1 \leq i \leq m$ and $\gamma_i \notin E$, for all i, $1 \leq i \leq n$. Thus, the rule

$$\frac{\alpha}{\gamma | \gamma_2 | \ldots | \gamma_n}$$

is in D^E. Since E' is closed under defaults in D^E, and $\gamma_i \notin E$, for all i, $1 \leq i \leq n$, it follows that if $\alpha \in E'$ then $\gamma \in E'$. Thus, E' is closed under the rule

$$\frac{\alpha}{\gamma}$$

and, more generally, under all rules from D'^E. Since E is an extension for D', it follows that $E = E'$. Thus, E is minimal closed under propositional consequence and rules in D^E. Consequently, E is an extension for D. □

Before we prove Theorems 6.2 and 6.3 we will consider one more equivalent way of defining extensions for a *ddt*. Let us consider a modal language $\mathcal{L}_L$ obtained by extending the language $\mathcal{L}$ with a single modal operator L. Operator L can loosely be interpreted as "is known" or "is believed". Consider a formula of the form

$$L\alpha \wedge \neg L\beta_1 \wedge \ldots \wedge \neg L\beta_m \Rightarrow L\gamma_1 \vee \ldots \vee L\gamma_n, \quad (15)$$

where α, β_i and γ_i are formulas from $\mathcal{L}$. Such formulas will be called *modal rules*, and collections of modal rules will be called *modal programs*. The modal rule (15) can be interpreted as:

> If α is known and it is not known that β_i ($\neg\beta_i$ is consistent), $1 \leq i \leq m$, then for some i, $1 \leq i \leq n$, γ_i is known.

Under such interpretation, modal rules are similar to disjunctive defaults. Below we show how to make this similarity precise.

The key notion is the concept of a *stable theory* ([Sta80, Moo85, MT91]. A theory $T \subseteq \mathcal{L}_L$ is stable if it satisfies the following three conditions:

ST1 $T = Cn(T)$,

ST2 If $\varphi \in T$ then $L\varphi \in T$,

ST3 If $\varphi \notin T$ then $\neg L\varphi \in T$.

Stable theories capture the intuition of belief sets of an agent with full introspection capabilities, and are of fundamental importance in nonmonotonic modal formalisms.

It is well-known ([Moo85, Mar89]) that for every theory $S \subseteq \mathcal{L}$ there is a unique stable theory T such that $T \cap \mathcal{L} = Cn(S)$. We denote this unique stable theory by $St(S)$.

Stable sets have the following property, which we will refer to as the *disjunctive property*: if T is stable and formulas φ and ψ are propositional combinations of formulas of the form $L\alpha$, where $\alpha \in \mathcal{L}$, then $\varphi \vee \psi \in T$ if and only if $\varphi \in T$ of $\psi \in T$ ([HM84, MT91]).

Stable sets can be ordered by the inclusion relation applied to their modal-free parts. Precisely, if T_1 and T_2 are stable, then $T_1 \sqsubseteq T_2$ if $T_1 \cap \mathcal{L} \subseteq T_2 \cap \mathcal{L}$. The purpose of this relation is to formalize the concept of minimal knowledge about the domain of interest.

Let I be a modal program and let $T \subseteq \mathcal{L}_L$ be a stable theory. In the next definition, we specify how to apply negation by failure to remove negation from the antecedents of the rules in I.

Definition 8.1 *The reduct of I with respect to T, I^T, is the modal program obtained from I by removing all rules with $\neg L\beta$ in the antecedent, if $\beta \in T$, and by removing the conjuncts $\neg L\beta$ from all other clauses.*

The next definition applies the minimal-knowledge paradigm to the reduct I^T.

Definition 8.2 *A stable theory T is called a modal extension for I if and only if T is a $\sqsubseteq$-minimal stable theory containing I^T.*

It follows from our informal comments earlier that a disjunctive default

$$d = \frac{\alpha : \beta_1, \ldots, \beta_m}{\gamma_1 | \ldots | \gamma_n}$$

can be interpreted by a modal rule

$$emb_L(d) = L\alpha \wedge \neg L\neg\beta_1 \ldots \neg L\neg\beta_m \Rightarrow L\gamma_1 \vee \ldots \vee L\gamma_n.$$

For a *ddt* D, by $emb_L(D)$ we denote the modal program obtained by replacing each default $d \in D$ with the rule $emb_L(d)$. We have the following theorem.

Theorem 8.3 *Let D be a ddt and let $S \subseteq \mathcal{L}$ be closed under propositional consequence. Theory S is an extension for D if and only if $St(S)$ is a modal extension for $emb_L(D)$.*

Proof: First notice that a rule

$$L\alpha \Rightarrow L\gamma_1 \vee \ldots \vee L\gamma_n$$

is in the reduct $emb_L(D)^{St(S)}$ if and only if the rule

$$\frac{\alpha}{\gamma_1|\ldots|\gamma_n}$$

is in the reduct D^S.

Next, observe that for any theory $S' \subseteq \mathcal{L}$ that is closed under propositional consequence,

$$emb_L(D)^{St(S)} \subseteq St(S')$$

if and only if S' is closed under the rules in D^S. Indeed, by the disjunctive property,

$$(L\alpha \Rightarrow L\gamma_1 \vee \ldots \vee L\gamma_n) \in St(S')$$

if and only if $\alpha \notin S'$, or at least one γ_i is in S'. Thus, the assertion follows by Theorem 5.3 and by the definition of a modal extension. □

Now we are ready to prove Theorems 6.2 and 6.3.

Theorem 6.2. *If E is an extension for D, then E is a minimal (with respect to inclusion) element in the class of all extensions for the covers of D.*

Proof: In the same way in which we defined a cover for a *ddt* D, we can define a *cover* of a modal program. By Theorem 8.3, it follows that to prove Theorem 6.2, it suffices to show that if a stable theory is a modal extension for a modal program I, then T is a modal extension for a cover of I.

First, assume that C_i, $1 \leq i \leq p$, are all the covers of I. It is easy to see that I is propositionally equivalent to the disjunction

$$\Phi_I = \Phi_1 \vee \ldots \vee \Phi_p,$$

where each Φ_I is a conjunction of formulas of the cover C_i. Each formula Φ_i is a propositional combination of formulas of the form $L\alpha$, where $\alpha \in \mathcal{L}$. Thus, the disjunctive property applies and if T is stable then $\Phi_I \in T$ if and only if $\Phi_i \in T$, for some i, $1 \leq i \leq p$.

Let us also observe that the reducts C_i^T, $1 \leq i \leq p$, are exactly the covers of the reduct I_T. It follows from the previous discussion that if U is stable, then $I^T \subseteq U$ if and only if $C_i^T \subseteq U$, for some i, $1 \leq i \leq p$.

Consider a modal extension T for the program I. Theory T is a $\sqsubseteq$-minimal stable theory containing I^T. Consequently, for some i, $C_i^T \subseteq T$. Let $U \sqsubseteq T$ be stable and such that $C_i^T \subseteq U$. Then $I^T \subseteq U$. Since T is $\sqsubseteq$-minimal with this property, $U = T$ and T is a $\sqsubseteq$-minimal stable theory such that $C_i^T \subseteq T$. Hence, T is a modal extension for C_i. □

Theorem 6.3. *If a ddt D consists only of justification-free defaults, then the classes of extensions for D and of minimal (with respect to inclusion) extensions for covers of D coincide.*

Proof: By Theorem 8.3 it follows that in order to prove Theorem 6.3, it suffices to show the following:

> If a modal program I consists entirely of the rules without conjuncts $\neg L\beta$ in the antecedent, then the classes of modal extensions for I and of $\sqsubseteq$-minimal extensions for covers of I coincide.

To this end, assume that $C_1, \ldots, C_p$ are all the covers if I. Observe that for any stable theory T, $I^T = I$ and $C_i^T = C_i$, $1 \leq i \leq p$. By Theorem 6.2, it suffices to prove only that a $\sqsubseteq$-minimal extension T for a cover of I is an extension for I. Clearly, $I^T = I \subseteq T$. Consider a stable theory $U \sqsubseteq T$ such that $I^T = I \subseteq U$. Then, as in the previous proof, we have that for some i, $C_i^T = C_i \subseteq U$. Since T is $\sqsubseteq$-minimal among all modal extensions for covers of I, it follows that $T = U$. Consequently, T is a modal extension for I. □

Theorem 7.2. *Let P be an extended disjunctive database. A set of literals M is an answer set of P if and only if M is the set of literals of an extension for $emb(P)$.*

Proof: Assume that a set of literals M is an answer set for P. Put $E = Cn(M)$. Consider an arbitrary default

$$\frac{a_1 \wedge \ldots \wedge a_k : \neg b_1, \ldots, \neg b_m}{c_1 \mid \ldots \mid c_n}$$

from $emb(P)$. Suppose that $a_1 \wedge \ldots \wedge a_k \in E$, and $\neg\neg\beta_i \notin E$, $1 \leq i \leq m$. Then, it follows that $a_i \in M$, $1 \leq i \leq k$, and $b_i \notin M$. Since M is an answer set for P, for some i, $1 \leq i \leq n$, $c_i \in M$. Hence, $c_i \in E$.

Let $E' \subseteq E$ be a theory closed under propositional consequence and such that for every default

$$\frac{a_1 \wedge \ldots \wedge a_k : \neg b_1, \ldots, \neg b_m}{c_1 \mid \ldots \mid c_n}$$

in $emb(P)$, if $a_i \in E'$, $1 \leq i \leq k$, and $\neg\neg b_i \notin E$, $1 \leq i \leq m$, then for some i, $1 \leq i \leq n$, $c_i \in E'$. Let M' be the set of literals in E'. Since E' is closed under propositional calculus, the condition 2 of Definition 7.1 holds. Consider now a rule

$$c_1 \mid \ldots \mid c_n \leftarrow a_1, \ldots, a_k, not\ b_1, \ldots, not\ b_m$$

from P. Assume that $a_i \in M'$, $1 \leq i \leq k$, and $b_i \notin M$, $1 \leq i \leq m$. Then, $a_i \in E'$, $1 \leq i \leq k$, and $\neg\neg b_i \notin E$, $1 \leq i \leq m$. Thus, for some i, $1 \leq i \leq n$, $c_i \in E'$. Consequently, $c_i \in M'$ and M' satisfies condition 1 of Definition 7.1. Since M is a minimal set satisfying this condition and $M' \subseteq M$, $M = M'$ follows. Thus, $E = E'$ follows, which implies that E is an extension for $emb(P)$.

The converse implication can be proved in the same manner. □

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