Infinitary Equilibrium Logic and Strongly Equivalent Logic Programs

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Abstract

Strong equivalence is an important concept in the theory of answer set programming. Informally speaking, two sets of rules are strongly equivalent if they have the same meaning in any context. Equilibrium logic was used to prove that sets of rules expressed as propositional formulas are strongly equivalent if and only if they are equivalent in the logic of here-and-there. We extend this line of work to formulas with infinitely long conjunctions and disjunctions, show that the infinitary logic of here-and-there characterizes strong equivalence of infinitary formulas, and give an axiomatization of that logic. This is useful because of the relationship between infinitary formulas and logic programs with local variables.

1. Introduction

Answer set programming (ASP) is a form of declarative programming based on the stable model semantics of logic programs [1, 2, 3, 4, 5, 6, 7]. The concept of strong equivalence plays an important role in the theory of ASP. Informally speaking, two sets of rules are strongly equivalent if they have the same meaning in any context.

Compare, for instance, the rules

$$q(X, Z) \leftarrow q(X, Y), \, q(Y, Z), \, p(X), \, p(Y), \, p(Z) \tag{1}$$

and

$$\leftarrow q(X, Y), \, q(Y, Z), \, not \, q(X, Z), \, p(X), \, p(Y), \, p(Z). \tag{2}$$

Both rules express the idea that relation q is transitive on domain p. But in many contexts these rules do not have the same meaning: the effect of adding (1)

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to a logic program describing p and q is, in general, not the same as the effect of adding (2). The first rule allows us to derive new facts about q; adding it to a program turns relation q into its transitive closure. The second rule is a constraint; adding it weeds out the stable models in which q is not transitive.

The situation is different, however, if the program to which we add rules (1) and (2) contains the choice rule

$$\{q(X, Y)\} \leftarrow p(X),\, p(Y) \tag{3}$$

("for any X, Y from p, decide arbitrarily whether to include $q(X, Y)$ in the stable model.") The set consisting of rules (1), (3) is strongly equivalent to the set consisting of (2), (3). Consequently, in the presence of choice rule (3) the program obtained by adding (1) has the same stable models as the program obtained by adding (2).

According to [8], strong equivalence is closely related to the 3-valued logic called the logic of here-and-there, which was introduced by Arend Heyting [9] long before the invention of computer programming.[1] Consider the ground instances of rules (1)–(3):

$$
\begin{aligned}
q(t_1, t_3) \quad &\leftarrow q(t_1, t_2),\, q(t_2, t_3),\, p(t_1),\, p(t_2),\, p(t_3),\\
&\leftarrow q(t_1, t_2),\, q(t_2, t_3),\, not\ q(t_1, t_3),\, p(t_1),\, p(t_2),\, p(t_3),\\
\{q(t_1, t_2)\} \quad &\leftarrow p(t_1),\, p(t_2)
\end{aligned}
$$

(t_1, t_2, t_3 are arbitrary ground terms) and rewrite these ground rules as propositional combinations of ground atoms in the following way:

$$q(t_1, t_2) \wedge q(t_2, t_3) \wedge p(t_1) \wedge p(t_2) \wedge p(t_3) \rightarrow q(t_1, t_3), \tag{4}$$

$$\neg(q(t_1, t_2) \wedge q(t_2, t_3) \wedge \neg q(t_1, t_3) \wedge p(t_1) \wedge p(t_2) \wedge p(t_3)), \tag{5}$$

$$p(t_1) \wedge p(t_2) \rightarrow q(t_1, t_2) \vee \neg q(t_1, t_2). \tag{6}$$

Formulas (4) and (5) are equivalent to each other in classical logic. But this fact cannot be established in the logic of here-and-there, which is weaker than classical logic. Formula (6) is a tautology; this fact cannot be established in the logic of here-and-there either. On the other hand, the equivalence between the set consisting of formulas of forms (4), (6) and the set consisting of formulas of forms (5), (6) can be proved even in this weaker logic. This example illustrates a general fact: two sets of rules written as propositional formulas are strongly equivalent if and only if they are equivalent in the logic of here-and-there [8, Theorem 1].

In view of this relationship, proving strong equivalence can be often reduced to reasoning in a system of axioms and inference rules that is sound and complete

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[1] The name "here-and-there" is appropriate in view of the fact that this logic can be described in terms of Kripke frames with two worlds, "Here" and "There." It is known also as "the logic of present and future" or "the Smetanich logic."

with respect to the logic of here-and-there. Such formal systems have been known for a long time; see Section 5.1.

The proof of the theorem relating strong equivalence to the logic of here-and-there is based on the characterization of stable models in terms of equilibrium logic [10]—a nonmonotonic counterpart of the logic of here-and-there.

The statement of the theorem is not restricted to finite sets of formulas. This is important because a single rule with variables has infinitely many ground instances if we allow function symbols (or symbols for arbitrary integers) in ground terms. But some rules found in ASP programs can be represented by sets of propositional formulas only if we allow formulas themselves to be infinite; infinite sets of finite formulas do not suffice. Consider, for instance, the rule

$$q \leftarrow count\{X \,:\, p(X)\} = 0. \tag{7}$$

The aggregate expression in the body means, informally speaking, that set p is empty. This rule can be thought of as an implication with an infinite conjunction in the body:

$$\bigwedge_t \neg p(t) \;\rightarrow\; q.$$

Here t ranges over ground terms. The need for infinite conjunctions and disjunctions is common when rules contain local variables, such as X in the example above. Many ASP programs, in particular many programs in the input language of the grounder GRINGO and its subset, the ASP Core language [11], can be represented by formulas of this type [12].

In many cases, first-order formulas can also be used to capture the meaning of ASP programs. For example, rule (7) can be represented using the first-order formula

$$\forall x \neg p(x) \;\rightarrow\; q.$$

But possibilities of this approach are more limited. For instance, if *count* in (7) is replaced with *sum*, or 0 is replaced by a variable, the resulting rule cannot be represented using a first-order formula.

In this paper, on the basis of the definition of a stable model for infinitary propositional formulas proposed by Miroslaw Truszczynski [13], we extend to such formulas some definitions and theorems of the theory of strong equivalence and equilibrium logic. Our goals are

(i) to define the infinitary version of the logic of here-and-there,

(ii) to define its nonmonotonic counterpart—the infinitary version of equilibrium logic,

(iii) to verify that stable models of infinitary formulas in the sense of Truszczynski can be characterized in terms of infinitary equilibrium logic,

(iv) to verify that infinitary propositional formulas are strongly equivalent to each other iff they are equivalent in the infinitary logic of here-and-there,

(v) to find an axiomatization of that logic.

We will see that achieving goals (i)–(iv) is straightforward, given the work done earlier for finite formulas. Goal (v) is more challenging.

A preliminary version of this paper was presented at the 2015 Conference on Logic Programming and Nonmonotonic Reasoning [14]. The material in Sections 2.4 and 5.3 is new to this version.

2. Stable Models and Equilibrium Logic in the Infinitary Setting

2.1. Review: Infinitary Formulas

Let Σ be a propositional signature, that is, a set of propositional atoms. The syntax of infinitary formulas defined by Truszczynski [13] can be described as follows. For every nonnegative integer r, *(infinitary propositional) formulas (over Σ) of rank r* are defined recursively, as follows:

- every atom from Σ is a formula of rank 0,

- if $\mathcal{H}$ is a set of formulas, and r is the smallest nonnegative integer that is greater than the ranks of all elements of $\mathcal{H}$, then $\mathcal{H}^{\wedge}$ and $\mathcal{H}^{\vee}$ are formulas of rank r,

- if F and G are formulas, and r is the smallest nonnegative integer that is greater than the ranks of F and G, then $F \to G$ is a formula of rank r.

We will write $\{F, G\}^{\wedge}$ as $F \wedge G$, and $\{F, G\}^{\vee}$ as $F \vee G$. The symbols $\top$ and $\bot$ will be understood as abbreviations for $\emptyset^{\wedge}$ and $\emptyset^{\vee}$ respectively; $\neg F$ stands for $F \to \bot$, and $F \leftrightarrow G$ stands for $(F \to G) \wedge (G \to F)$. These conventions allow us to view finite propositional formulas over Σ as a special case of infinitary formulas.

A set or family of formulas is *bounded* if the ranks of its members are bounded from above. For any bounded family $(F_\alpha)_{\alpha \in A}$ of formulas, we denote the formula $\{F_\alpha : \alpha \in A\}^{\wedge}$ by $\bigwedge_{\alpha \in A} F_\alpha$, and similarly for disjunctions.

For example, if $p_1, p_2, \ldots$ and q are atoms then each $\neg p_i$ is a formula of rank 1, $\bigwedge_i \neg p_i$ is a formula of rank 2, and

$$\bigwedge_i \neg p_i \ \to \ q \tag{8}$$

is a formula of rank 3.

Subsets of a signature Σ will be also called *interpretations* of Σ. The satisfaction relation between an interpretation and a formula is defined recursively, as follows:

- For every atom p from Σ, $I \models p$ if $p \in I$.

- $I \models \mathcal{H}^{\wedge}$ if for every formula F in $\mathcal{H}$, $I \models F$.

- $I \models \mathcal{H}^{\vee}$ if there is a formula F in $\mathcal{H}$ such that $I \models F$.

- $I \models F \to G$ if $I \not\models F$ or $I \models G$.

A *model* of a set $\mathcal{H}$ of infinitary formulas is an interpretation that satisfies all formulas in $\mathcal{H}$. A formula is *tautological* if it is satisfied by all interpretations.

2.2. Review: Stable Models of Infinitary Formulas

Truszczynski's definition of the reduct of an infinitary formula is a straightforward extension of the definition proposed by Paolo Ferraris [15] for the finite case. The *reduct* F^I of a formula F with respect to an interpretation I is defined recursively:

- For every atom p from Σ, p^I is p if $p \in I$, and $\perp$ otherwise.

- $(\mathcal{H}^\wedge)^I = \{G^I \mid G \in \mathcal{H}\}^\wedge$.

- $(\mathcal{H}^\vee)^I = \{G^I \mid G \in \mathcal{H}\}^\vee$.

- $(G \to H)^I$ is $G^I \to H^I$ if $I \models G \to H$, and $\perp$ otherwise.

Let us calculate, for instance, the reduct of formula (8) with respect to the interpretation $\{p_1\}$. The reduct of $\neg p_i$ is $\perp$ if $i = 1$ and $\neg\perp$ otherwise. Consequently the reduct of the antecedent of (8) is $\perp \wedge \neg\perp$, and the reduct of (8) is the tautology

$$\perp \wedge \neg\perp \to \perp. \tag{9}$$

An interpretation I is a *stable model* of a set $\mathcal{H}$ of formulas if it is minimal with respect to set inclusion among the interpretations satisfying the reducts F^I of all formulas F from $\mathcal{H}$.

For example, $\{p_1\}$ is a stable model of the set

$$\left\{ p_1, \ \bigwedge_i \neg p_i \ \to \ q \right\} \tag{10}$$

because $\{p_1\}$ is minimal among the interpretations satisfying p_1 and (9).

2.3. Infinitary Logic of Here-and-There

An *HT-interpretation* of Σ is an ordered pair $\langle I, J \rangle$ of interpretations of Σ such that $I \subseteq J$. Intuitively, the atoms in I are true "here" ("in the world H"), and the atoms in J are true "there" ("in the world T").

The satisfaction relation between an HT-interpretation and a formula is defined recursively, as follows:

- For every atom p from Σ, $\langle I, J \rangle \models p$ if $p \in I$.

- $\langle I, J \rangle \models \mathcal{H}^\wedge$ if for every formula F in $\mathcal{H}$, $\langle I, J \rangle \models F$.

- $\langle I, J \rangle \models \mathcal{H}^\vee$ if there is a formula F in $\mathcal{H}$ such that $\langle I, J \rangle \models F$.

- $\langle I, J \rangle \models F \to G$ if

  (i) $\langle I, J \rangle \not\models F$ or $\langle I, J \rangle \models G$, and
  (ii) $J \models F \to G$.

An *HT-model* of a set $\mathcal{H}$ of infinitary formulas is an HT-interpretation that satisfies all formulas in $\mathcal{H}$.

Let us check, for instance, that the HT-interpretation $\langle \emptyset, \{p\} \rangle$ satisfies the formula $\neg\neg p$. According to the clause for implication, it is sufficient to check that $\langle \emptyset, \{p\} \rangle \not\models \neg p$ and $\{p\} \models \neg\neg p$. The second condition is obvious; to check the first, observe that $\{p\} \not\models \neg p$.

The semantics of infinitary formulas defined above can be viewed as a 3-valued semantics as follows. About a formula F we say that it is *forced in the world H* of an HT-interpretation $\langle I, J \rangle$ if it is satisfied by $\langle I, J \rangle$; we will say that it is *forced in the world T* if it is satisfied by J. The set of worlds in which F is forced will be called the *truth value* of F with respect to $\langle I, J \rangle$. It is easy to check by induction on the rank that every formula that is forced in H is forced in T as well. Consequently, the only possible truth values of a formula are $\emptyset$, $\{T\}$, and $\{H, T\}$.

2.4. Negative Formulas

According to Glivenko's theorem [16], every tautology that begins with negation is intuitionistically valid. Mark Nadel [17] observed that this theorem does not generalize to infinitary propositional formulas. In this section we prove a generalization of the Glivenko property for the infinitary logic of here-and-there.

The class of *negative* formulas is defined recursively:

- $\mathcal{H}^\wedge$ and $\mathcal{H}^\vee$ are negative if every formula in $\mathcal{H}$ is negative;

- $F \to G$ is negative if G is negative.

It is clear, for instance, that $\top$, $\bot$, and all formulas of the forms $\neg F$ and $F \to \neg G$ are negative.

Theorem 1. *Every negative tautological formula is satisfied by all HT-interpretations.*

This fact shows, for instance, that the infinitary De Morgan laws

$$\neg \bigwedge_\alpha F_\alpha \leftrightarrow \bigvee_\alpha \neg F_\alpha,$$

$$\neg \bigvee_\alpha F_\alpha \leftrightarrow \bigwedge_\alpha \neg F_\alpha$$

are satisfied by all HT-interpretations.

Theorem 1 is immediate from the following lemma.

Lemma 1. *If a formula F is negative then an HT-interpretation $\langle I, J \rangle$ satisfies F iff J satisfies F.*

Proof. The only-if part holds for all formulas, not only those that are negative. We will prove the if part by induction on the definition of negative formulas.

The case of $\mathcal{H}^\wedge$ and $\mathcal{H}^\vee$ is straightforward. Assume that $J \models F \to G$ and G is negative. We need to check that

$$\langle I, J \rangle \not\models F \text{ or } \langle I, J \rangle \models G.$$

If $J \not\models F$ then $\langle I, J \rangle \not\models F$. Otherwise $J \models G$ and by the induction hypothesis it follows that $\langle I, J \rangle \models G$.

2.5. Equilibrium Models of Infinitary Formulas

An HT-interpretation $\langle I, J \rangle$ is *total* if $I = J$. It is clear that a total HT-interpretation $\langle J, J \rangle$ satisfies F iff J satisfies F.

An *equilibrium model* of a set $\mathcal{H}$ of infinitary formulas is a total HT-model $\langle J, J \rangle$ of $\mathcal{H}$ such that for every proper subset I of J, $\langle I, J \rangle$ is not an HT-model of $\mathcal{H}$.

The following proposition is similar to Theorem 1 from [15].

Theorem 2. *An interpretation J is a stable model of a set $\mathcal{H}$ of infinitary formulas iff $\langle J, J \rangle$ is an equilibrium model of $\mathcal{H}$.*

For example, $\langle \{p_1\}, \{p_1\} \rangle$ is an equilibrium model of (10). Indeed, the interpretation $\{p_1\}$ satisfies both formulas from (10), and the HT-interpretation $\langle \emptyset, \{p_1\} \rangle$ does not satisfy the first of them.

Lemma 2. *For any infinitary formula F and any HT-interpretation $\langle I, J \rangle$,*

$$I \models F^J \text{ iff } \langle I, J \rangle \models F.$$

The lemma can be proved by strong induction on the rank of F.

Proof of Theorem 2. It follows from the lemma that a total HT-interpretation $\langle J, J \rangle$ is an equilibrium model of $\mathcal{H}$ iff

- J satisfies all formulas from $\mathcal{H}$, and

- there is no proper subset I of J such that I satisfies the reducts F^J of all formulas F from $\mathcal{H}$.

These conditions express that J is a stable model of $\mathcal{H}$.

3. Strong Equivalence in the Infinitary Setting

3.1. Strong Equivalence and HT-models

About sets $\mathcal{H}_1$, $\mathcal{H}_2$ of infinitary formulas we say that they are *strongly equivalent* to each other if, for every set $\mathcal{H}$ of infinitary formulas, the sets $\mathcal{H}_1 \cup \mathcal{H}$ and $\mathcal{H}_2 \cup \mathcal{H}$ have the same stable models. About formulas F and G we say that they are *strongly equivalent* if the singleton sets $\{F\}$ and $\{G\}$ are strongly equivalent.

A *unary formula* is an atom or a formula of the form $p \to q$, where p and q are atoms. The following theorem is similar to the main theorem from [8].

Theorem 3. *For any sets $\mathcal{H}_1$, $\mathcal{H}_2$ of infinitary formulas, the following conditions are equivalent:*

(i) *$\mathcal{H}_1$ is strongly equivalent to $\mathcal{H}_2$,*
(ii) *for every set $\mathcal{H}$ of unary formulas, sets $\mathcal{H}_1 \cup \mathcal{H}$ and $\mathcal{H}_2 \cup \mathcal{H}$ have the same stable models;*
(iii) *sets $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same HT-models.*

For instance, the formulas $F \vee \neg F$ and $\neg\neg F \rightarrow F$, where F is an arbitrary infinitary formula, are strongly equivalent to each other. Indeed, an HT-interpretation does *not* satisfy $F \vee \neg F$ iff the truth value of F with respect to that interpretation is $\{T\}$, and the same holds for $\neg\neg F \rightarrow F$.

Proof of Theorem 3. Clearly, (i) implies (ii). To see that (iii) implies (i), observe that if sets $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same HT-models then $\mathcal{H}_1 \cup \mathcal{H}$ and $\mathcal{H}_2 \cup \mathcal{H}$ have the same HT-models, and consequently have the same equilibrium models. It follows by Theorem 2 that $\mathcal{H}_1 \cup \mathcal{H}$ and $\mathcal{H}_2 \cup \mathcal{H}$ have the same stable models.

It remains to check that (ii) implies (iii). Suppose $\langle I, J \rangle$ is an HT-model of $\mathcal{H}_1$ but not an HT-model of $\mathcal{H}_2$. We will show how to find a set $\mathcal{H}$ of unary formulas such that $\langle J, J \rangle$ is an equilibrium model of one of the sets $\mathcal{H}_1 \cup \mathcal{H}, \mathcal{H}_2 \cup \mathcal{H}$ but not the other. It will follow that the interpretation J is a stable model of one but not the other.

Case 1: $\langle J, J \rangle$ is not an HT-model of $\mathcal{H}_2$. Since $\langle I, J \rangle$ is an HT-model of $\mathcal{H}_1$, it is easy to see that $\langle J, J \rangle$ must be an HT-model of $\mathcal{H}_1$ as well. Then we can take $\mathcal{H} = J$. Indeed, it is clear that $\langle J, J \rangle$ is an HT-model of $\mathcal{H}_1 \cup J$. Furthermore, for any I that is a proper subset of J, $\langle I, J \rangle$ cannot be an HT-model of $\mathcal{H}_1 \cup J$, so that $\langle J, J \rangle$ is an equilibrium model of $\mathcal{H}_1 \cup J$. On the other hand, since $\langle J, J \rangle$ is not a HT-model of $\mathcal{H}_2$, it cannot be an HT-model of $\mathcal{H}_2 \cup J$.

Case 2: $\langle J, J \rangle$ is an HT-model of $\mathcal{H}_2$. Let $\mathcal{H}$ be the set

$$I \cup \{p \rightarrow q \mid p, q \in J \setminus I\}.$$

Since $\langle J, J \rangle$ satisfies every formula in $\mathcal{H}$, it is an HT-model of $\mathcal{H}_2 \cup \mathcal{H}$. To see that it is an equilibrium model, consider any HT-model $\langle K, J \rangle$ of $\mathcal{H}_2 \cup \mathcal{H}$. Clearly, K must contain I. But it cannot be equal to I, since $\langle I, J \rangle$ is not an HT-model of $\mathcal{H}_2$. Thus $I \subset K \subset J$. Consider an atom p in $K \setminus I$ and an atom q in $J \setminus K$. For these atoms, $p \rightarrow q$ belongs to $\mathcal{H}$. But $\langle K, J \rangle$ does not satisfy this implication, contrary to the assumption that it is an HT-model of $\mathcal{H}_2 \cup \mathcal{H}$. We may conclude that $\langle J, J \rangle$ is an equilibrium model of $\mathcal{H}_2 \cup \mathcal{H}$. Finally, we will check that $\langle J, J \rangle$ is not an equilibrium model of $\mathcal{H}_1 \cup \mathcal{H}$. Consider the HT-model $\langle I, J \rangle$ of $\mathcal{H}_1$. Clearly, it is an HT-model of I. Moreover, it satisfies each implication $p \rightarrow q$ in $\mathcal{H}$: $\langle I, J \rangle$ does not satisfy p because $p \notin I$, and J satisfies q because $q \in J$. We see that $\langle I, J \rangle$ satisfies all formulas in $\mathcal{H}$, so that it is an HT-model of $\mathcal{H}_1 \cup \mathcal{H}$. Furthermore, I is different from J since $\langle J, J \rangle$ is an HT-model of $\mathcal{H}_2$ and $\langle I, J \rangle$ is not. Consequently, I is a proper subset of J, and we may conclude that $\langle J, J \rangle$ is not an equilibrium model of $\mathcal{H}_1 \cup \mathcal{H}$.

3.2. Substitutions

A part of any formula can be replaced with a strongly equivalent formula without changing the set of stable models. For instance, it is easy to check that the formulas $p \wedge \neg p$ and $\bot$ are strongly equivalent to each other; it follows that the formulas

$$F \wedge (q \rightarrow (p \wedge \neg p)) \qquad \text{and} \qquad F \wedge \neg q \qquad (11)$$

have the same stable models. Corollary 1 below expresses a more general fact: several parts (even infinitely many) can be simultaneously replaced by strongly equivalent formulas. Its statement uses the following definitions, based on Harrison *et al.* [18].

Let Σ and Σ' be propositional signatures. A *substitution (from Σ' to Σ)* is a function ϕ that maps each atom from Σ' to an infinitary formula over Σ, such that the range of ϕ is bounded. A substitution is extended from the atoms of Σ' to arbitrary infinitary formulas over Σ' as follows:

- If F is $\mathcal{H}^{\wedge}$ then $\phi F = \{\phi G \mid G \in \mathcal{H}\}^{\wedge}$.

- If F is $\mathcal{H}^{\vee}$ then $\phi F = \{\phi G \mid G \in \mathcal{H}\}^{\vee}$.

- If F is $G \rightarrow H$ then $\phi F = \phi G \rightarrow \phi H$.

Consider, for instance, a pair of formulas (11) of a signature Σ. Let Σ' be $\Sigma \cup \{r\}$, where r is a new atom. If ϕ is the function that maps r to $p \wedge \neg p$ and all other atoms to themselves, and ψ is the function that maps r to $\bot$ and all other atoms to themselves, then the formulas (11) are $\phi(F \wedge (q \rightarrow r))$ and $\psi(F \wedge (q \rightarrow r))$.

Corollary 1. *Let ϕ and ψ be substitutions from Σ' to Σ such that for all $p \in \Sigma'$, ϕp is strongly equivalent to ψp. Then for any formula F over Σ', ϕF is strongly equivalent to ψF, so that ϕF and ψF have the same stable models.*

Proof. By Theorem 3, the assertion of the corollary can be stated as follows: if for all $p \in \Sigma'$, ϕp and ψp are satisfied by the same HT-interpretations, then for any formula F over Σ', ϕF and ψF are satisfied by the same HT-interpretations. This is easy to check by induction on the rank of F.

4. Axiomatization of the Infinitary Logic of Here-and-There

We will first discuss the easier problem of axiomatizing classical infinitary logic [19], [18, Remark 4].

4.1. Review: Axioms and Inference Rules for Classical Infinitary Logic

The derivable objects in the deductive system C^{∞} are *(infinitary) sequents*—expressions of the form $\Gamma \Rightarrow F$, where F is an infinitary formula, and Γ is a finite set of infinitary formulas ("F under assumptions Γ"). To simplify notation, we will write Γ as a list. We will identify a sequent of the form $\Rightarrow F$ with the formula F.

The inference rules are the introduction and elimination rules for the propositional connectives and the weakening rule, as shown in Table 4.1.

$$(\wedge I) \ \frac{\Gamma \Rightarrow H \quad \text{for all } H \in \mathcal{H}}{\Gamma \Rightarrow \mathcal{H}^{\wedge}} \qquad (\wedge E) \ \frac{\Gamma \Rightarrow \mathcal{H}^{\wedge}}{\Gamma \Rightarrow H} \quad (H \in \mathcal{H})$$

$$(\vee I) \ \frac{\Gamma \Rightarrow H}{\Gamma \Rightarrow \mathcal{H}^{\vee}} \quad (H \in \mathcal{H}) \qquad (\vee E) \ \frac{\Gamma \Rightarrow \mathcal{H}^{\vee} \quad \Delta, H \Rightarrow F \quad \text{for all } H \in \mathcal{H}}{\Gamma, \Delta \Rightarrow F}$$

$$(\to I) \ \frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \to G} \qquad (\to E) \ \frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow F \to G}{\Gamma, \Delta \Rightarrow G}$$

$$(W) \ \frac{\Gamma \Rightarrow F}{\Gamma, \Delta \Rightarrow F}.$$

Table 1: Inference Rules of C^{∞}. In the introduction and elimination rules for conjunction and disjunction, $\mathcal{H}$ is a bounded set of formulas.

The axiom schemas of C^{∞} are
$$F \Rightarrow F$$
and
$$\bigvee_{B \subseteq A} \left(\bigwedge_{\alpha \in B} F_{\alpha} \wedge \bigwedge_{\alpha \in A \setminus B} \neg F_{\alpha} \right) \tag{12}$$
where $\{F_{\alpha}\}_{\alpha \in A}$ is a non-empty bounded family of formulas. The latter generalizes the law of the excluded middle: if $A = \{1\}$ then (12) becomes

$$(F_1 \wedge \top) \vee (\top \wedge \neg F_1).$$

The need for this generalization is discussed in Section 5.3. The set of *theorems of C^{∞}* is the smallest set of sequents that includes the axioms of the system and is closed under the application of its inference rules.

System C^{∞} is sound and complete:

Theorem 4. *An infinitary formula is a theorem of C^{∞} iff it is tautological.*

The soundness part is straightforward. The proof of completeness, outlined in [18, Footnote 4], is analogous to the proof of completeness for the finite case due to László Kalmár [20]. It can be presented using the following notation: for any interpretation I, M_I stands for the set

$$I \cup \{\neg p \mid p \in \Sigma \setminus I\}.$$

It is easy to check by induction that for any formula F, $M_I^{\wedge} \to F$ is a theorem of C^{∞} if I satisfies F, and $M_I^{\wedge} \to \neg F$ is a theorem of C^{∞} otherwise. In particular, if F is tautological then $M_I^{\wedge} \to F$ is a theorem of C^{∞} for any interpretation I. On the other hand, the disjunction of the formulas $M_I^{\wedge}$ over all interpretations I is an instance of axioms schema (12): take $\{F_{\alpha}\}_{\alpha \in A}$ to be the family of all atoms. It follows that every tautological formula is a theorem.

4.2. Axioms and Inference Rules for the Infinitary Logic of Here-and-There

The infinitary deductive system HT^∞ is obtained from C^∞ by replacing the generalized law of the excluded middle (12) with two axiom schemas:

$$F \vee (F \to G) \vee \neg G \tag{13}$$

and

$$\bigwedge_{\alpha \in A} \bigvee_{F \in \mathcal{H}_\alpha} F \to \bigvee_{(F_\alpha)_{\alpha \in A}} \bigwedge_{\alpha \in A} F_\alpha \tag{14}$$

for every non-empty family $(\mathcal{H}_\alpha)_{\alpha \in A}$ of sets of formulas such that its union is bounded; the disjunction in the consequent of (14) extends over all elements $(F_\alpha)_{\alpha \in A}$ of the Cartesian product of the family $(\mathcal{H}_\alpha)_{\alpha \in A}$. Axiom schema (14) generalizes one direction of the distributivity of conjunction over disjunction to infinitary formulas: if $A = \{1, 2\}$, $\mathcal{H}_1 = \{F_1, G_1\}$, and $\mathcal{H}_2 = \{F_2, G_2\}$, then (14) turns into

$$(F_1 \vee G_1) \wedge (F_2 \vee G_2) \to (F_1 \wedge F_2) \vee (F_1 \wedge G_2) \vee (G_1 \wedge F_2) \vee (G_1 \wedge G_2).$$

We say that formulas F and G are *equivalent in HT^∞* if $F \leftrightarrow G$ is a theorem of HT^∞.

The following theorem expresses the soundness and completeness of HT^∞.

Theorem 5. *An infinitary formula is a theorem of HT^∞ iff it is satisfied by all HT-interpretations.*

The proof is given in Section 4.3.

From Theorems 3 and 5 we conclude:

Corollary 2. *Bounded sets $\mathcal{H}_1$, $\mathcal{H}_2$ of infinitary formulas are strongly equivalent iff $\mathcal{H}_1^\wedge$ is equivalent to $\mathcal{H}_2^\wedge$ in HT^∞.*

4.3. Proof of Theorem 5

The proof applies the idea of the proof of Theorem 4 above to the 3-valued case. Soundness is straightforward. The proof of completeness uses the following construction, due to Pedro Cabalar and Paolo Ferraris [21, Section 5]: for any HT-interpretation $\langle I, J \rangle$, M_{IJ} stands for

$$I \cup \{\neg\neg p \mid p \in J\} \cup \{\neg p \mid p \in \Sigma \setminus J\} \cup \{p \to q \mid p, q \in J \setminus I\}.$$

By $v_{IJ}(F)$ we denote the truth value of F with respect to $\langle I, J \rangle$ (see Section 2.3). We will omit the subscripts I, J in M_{IJ} and $v_{IJ}(F)$ when it is clear which HT-interpretation we refer to.

Lemma 3. *For any infinitary formula F and HT-interpretation $\langle I, J \rangle$,*

   (i) if $v(F) = \emptyset$ then $M^\wedge \Rightarrow \neg F$ is a theorem of HT^∞;

   (ii) if $v(F) = \{T\}$ then for every atom q in $J \setminus I$, $M^\wedge \Rightarrow F \leftrightarrow q$ is a theorem of HT^∞;

(iii) if $v(F) = \{H, T\}$ then $M^\wedge \Rightarrow F$ is a theorem of HT^∞.

Proof. We will prove the claim by strong induction on the rank of F. We assume the claim holds for all formulas with rank less than n and show that it holds for a formula F of rank n. We consider cases corresponding to the different possible forms of F and truth values $v(F)$. Note that if $v(F)$ is $\{T\}$ then the set $J \setminus I$ is non-empty. Indeed, if $I = J$ then the truth value of any formula is either $\emptyset$ or $\{H, T\}$.

Case 1: F is an atom.

Case 1.1: $v(F) = \emptyset$. Then $F \in \Sigma \setminus J$, and $\neg F \in M$.

Case 1.2: $v(F) = \{T\}$. Then $F \in J \setminus I$, and for every atom q in $J \setminus I$, the implications $F \to q$ and $q \to F$ are in M.

Case 1.3: $v(F) = \{H, T\}$. Then $F \in M$.

Case 2: F is of the form $\mathcal{H}^\wedge$. The induction hypothesis is then applicable to all formulas in $\mathcal{H}$.

Case 2.1: $v(F) = \emptyset$. Then there exists a formula G in $\mathcal{H}$ such that $v(G)$ is $\emptyset$. By the induction hypothesis, $M^\wedge \Rightarrow \neg G$ is a theorem of HT^∞. From this we can derive $M^\wedge \Rightarrow \neg(\mathcal{H}^\wedge)$.

Case 2.2: $v(F) = \{T\}$. Let $\mathcal{H}_1$ be the set of all formulas in $\mathcal{H}$ with truth value $\{T\}$, and $\mathcal{H}_2$ be the set of all formulas in $\mathcal{H}$ with truth value $\{H, T\}$. It is clear that $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{H}$ and that $\mathcal{H}_1$ is non-empty. Consider an arbitrary element q of $J \setminus I$. By the induction hypothesis $M^\wedge \Rightarrow G \leftrightarrow q$ is a theorem for every G in $\mathcal{H}_1$, and $M^\wedge \Rightarrow G$ is a theorem for every G in $\mathcal{H}_2$. From these we can derive $M^\wedge \Rightarrow \mathcal{H}_1^\wedge \leftrightarrow q$ and $M^\wedge \Rightarrow \mathcal{H}_2^\wedge$. Then we can derive $M^\wedge \Rightarrow \mathcal{H}^\wedge \leftrightarrow q$.

Case 2.3: $v(F) = \{H, T\}$. Then for each element G in $\mathcal{H}$, $v(G) = \{H, T\}$, and by the induction hypothesis $M^\wedge \Rightarrow G$ is a theorem. From these sequents we can derive $M^\wedge \Rightarrow \mathcal{H}^\wedge$.

Case 3: F is of the form $\mathcal{H}^\vee$. The induction hypothesis is then applicable to all formulas in $\mathcal{H}$.

Case 3.1: $v(F) = \emptyset$. Then for each element G in $\mathcal{H}$, $v(G) = \emptyset$, and by the induction hypothesis $M^\wedge \Rightarrow \neg G$ is a theorem. From these sequents we can derive $M^\wedge \Rightarrow \neg(\mathcal{H}^\vee)$.

Case 3.2: $v(F) = \{T\}$. Let $\mathcal{H}_1$ be the set of all formulas in $\mathcal{H}$ with truth value $\{T\}$, and $\mathcal{H}_2$ be the set of all formulas in $\mathcal{H}$ with truth value $\emptyset$. It is clear that $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{H}$ and that $\mathcal{H}_1$ is non-empty. Consider an arbitrary element q of $J \setminus I$. By the induction hypothesis $M^\wedge \Rightarrow G \leftrightarrow q$ is a theorem for every G in $\mathcal{H}_1$, and $M^\wedge \Rightarrow \neg G$ is a theorem for every G in $\mathcal{H}_2$. From these we can derive $M^\wedge \Rightarrow \mathcal{H}_1^\vee \leftrightarrow q$ and $M^\wedge \Rightarrow \neg(\mathcal{H}_2^\vee)$. Then we can derive $M^\wedge \Rightarrow \mathcal{H}^\vee \leftrightarrow q$.

Case 3.3: $v(F) = \{H, T\}$. Then there exists a formula G in $\mathcal{H}$ such that $v(G)$ is $\{H, T\}$. By the induction hypothesis, $M^\wedge \Rightarrow G$ is a theorem. From this we can derive $M^\wedge \Rightarrow \mathcal{H}^\vee$.

Case 4: F is of the form $F_1 \to F_2$. The induction hypothesis is then applicable to F_1 and F_2.

Case 4.1: $v(F) = \emptyset$. Then $v(F_1)$ is non-empty and $v(F_2)$ is empty.

Case 4.1.1: $v(F_1) = \{T\}$. By the induction hypothesis $M^\wedge \Rightarrow \neg F_2$ is a theorem, as is $M^\wedge \Rightarrow F_1 \leftrightarrow q$ for any q in $J \setminus I$. Consider an atom q in $J \setminus I$. By the construction of M, we know that $\neg\neg q$ is an element of M. From the sequents $M^\wedge \Rightarrow F_1 \leftrightarrow q$, $M^\wedge \Rightarrow \neg F_2$, and $M^\wedge \Rightarrow \neg\neg q$, we can derive $M^\wedge \Rightarrow \neg(F_1 \to F_2)$.

Case 4.1.2: $v(F_1) = \{H, T\}$. By the induction hypothesis, both $M^\wedge \Rightarrow F_1$ and $M^\wedge \Rightarrow \neg F_2$ are theorems. From these sequents we can derive $M^\wedge \Rightarrow \neg(F_1 \to F_2)$.

Case 4.2: $v(F) = \{T\}$. Then $v(F_1) = \{H, T\}$ and $v(F_2) = \{T\}$. By the induction hypothesis $M^\wedge \Rightarrow F_2 \leftrightarrow q$ is a theorem for any $q \in J \setminus I$, and $M^\wedge \Rightarrow F_1$ is a theorem as well. From these two sequents we can derive $M^\wedge \Rightarrow (F_1 \to F_2) \leftrightarrow q$.

Case 4.3: $v(F) = \{H, T\}$.

Case 4.3.1: $v(F_1) = \emptyset$. Then by the induction hypothesis $M^\wedge \Rightarrow \neg F_1$ is a theorem. From this we can derive $M^\wedge \Rightarrow F_1 \to F_2$.

Case 4.3.2: $v(F_2) = \{H, T\}$. Then by the induction hypothesis $M^\wedge \Rightarrow F_2$ is a theorem. From this we can derive $M^\wedge \Rightarrow F_1 \to F_2$.

Case 4.3.3: $v(F_1) \neq \emptyset$ and $v(F_2) \neq \{H, T\}$. Since $v(F)$ is $\{H, T\}$, $v(F_1)$ is different from $\{H, T\}$ and therefore must be equal to $\{T\}$. It follows that $v(F_2)$ is different from $\emptyset$, and therefore must be $\{T\}$ also. Consider an element q in $J \setminus I$. By the induction hypothesis both $M^\wedge \Rightarrow F_1 \leftrightarrow q$ and $M^\wedge \Rightarrow F_2 \leftrightarrow q$ are theorems. From these two sequents we can derive $M^\wedge \Rightarrow F_1 \to F_2$.

Note that in the proof of the lemma we did not refer to axiom schemas (13) and (14); the assertion of the lemma would hold even if those axioms were removed from HT^∞.

Lemma 4. *The disjunction of the formulas $M^\wedge_{IJ}$ over all HT-interpretations $\langle I, J \rangle$ is a theorem of HT^∞.*

Proof. Let Q stand for the set of disjunctions

$$p \vee (p \to q) \vee \neg q, \tag{15}$$

$$\neg p \vee \neg\neg p \tag{16}$$

for all p, q from Σ. Formulas of the form (16) are instances of the "weak law of excluded middle", and it is easy to verify that they (like formulas (15)) are theorems of HT^∞. Let $(\mathcal{H}_D)_{D \in Q}$ be the following family of sets:

$$\mathcal{H}_D = \{p, p \to q, \neg q\} \quad \text{if } D = p \vee (p \to q) \vee \neg q;$$
$$\mathcal{H}_D = \{\neg p, \neg\neg p\} \quad \text{if } D = \neg p \vee \neg\neg p.$$

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Then the formula

$$\bigwedge_{D\in Q}\bigvee_{S\in\mathcal{H}_D} S \to \bigvee_{(S_D)_{D\in Q}}\bigwedge_{D\in Q} S_D,$$

(where the disjunction in the consequent extends over all elements $(S_D)_{D\in Q}$ of the Cartesian product of the family $(\mathcal{H}_D)_{D\in Q}$) is an instance of axiom schema (14). Since the antecedent of this implication is the conjunction of all formulas in Q, it is a theorem of HT^∞. It follows that the consequent is a theorem as well. To complete the proof it is sufficient to show that for every disjunctive term

$$\bigwedge_{D\in Q} S_D \tag{17}$$

of the consequent there exists an HT-interpretation $\langle I,J\rangle$ such that the sequent

$$\bigwedge_{D\in Q} S_D \Rightarrow M_{IJ}^\wedge \tag{18}$$

is a theorem.

Consider one of the conjunctions (17), and let C be set of its conjunctive terms. The elements of C are formulas of the forms

$$p,\ \neg p,\ \neg\neg p,\ p\to q.$$

If C contains both a formula and its negation then (18) is a theorem for every $\langle I,J\rangle$. Otherwise, let I denote the set of all atoms in C, and J denote the set of all atoms p such that $\neg\neg p$ is in C. Let us check that $I\subseteq J$. Assume $p\in I$ so that $p\in C$. Since C is consistent, it does not contain $\neg p$, and since it contains a term from each disjunction (16), it contains $\neg\neg p$. So $\langle I,J\rangle$ is an HT-interpretation.

We will show that every formula from M_{IJ} belongs to C. By the choice of I, $I\subseteq C$. By the choice of J, $\{\neg\neg p\mid p\in J\}\subseteq C$. Consequently $\{\neg p\mid p\in\Sigma\setminus J\}\subseteq C$, because C contains one term from each disjunction (16). Finally, we need to check that $\{p\to q\mid p,q\in J\setminus I\}\subseteq C$. Consider a pair of atoms p,q that occur in J but not in I. By the choice of I, p is not in C, and by the choice of J, $\neg q$ is not in C. Since C contains one term from each of the disjunctions (15) and contains neither p nor $\neg q$, C must contain $p\to q$.

Proof of Completeness. Let F be an infinitary formula over signature Σ that is satisfied by all HT-interpretations of Σ. By Lemma 3(iii), $M_{IJ}\Rightarrow F$ is a theorem of HT^∞ for all HT-interpretations $\langle I,J\rangle$. By Lemma 4, it follows that F is a theorem also.

It is clear from the proof that HT^∞ will remain complete if we require that formulas F and G in axiom schema (13) be literals and that the sets $\mathcal{H}_i$ in axiom schema (14) be finite.

5. Related Work

5.1. Axiomatizations of the Logic of Here-and-There

The first axiomatization of the logic of here-and-there was given without proof by Jan Łukasiewicz [22]: add the axiom schema

$$(\neg F \to G) \to (((G \to F) \to G) \to G)) \tag{19}$$

to propositional intuitionistic logic. This axiomatization was rediscovered and proved complete by Ivo Thomas [23]. (In the notation of that paper, schema (19) is $3_2''$.) The simpler axiom schema (13) was proposed by Toshio Umezawa [24], and the completeness of the system obtained by adding that schema to intuitionistic logic was proved by Tsutomu Hosoi [25].

5.2. Early Work on Infinitary Formulas

Truszczynski's definition of an infinitary propositional formula reviewed in Section 2.1 differs from the syntax of infinitary formulas defined by Dana Scott and Alfred Tarski [19] and Carol Karp [26] in two ways. On the one hand, it is more restrictive, because it requires the ranks of formulas to be finite, and forming the conjunction or disjunction of a set of formulas is allowed only if the set is bounded. On the other hand, it does not restrict the cardinality of this set. Let ξ be the cardinality of the underlying infinite signature. Then we can form 2^ξ formulas of rank 1. A conjunction of rank 2 can have up to 2^ξ conjunctive terms, so that we can form up to 2^{2^ξ} formulas of rank 2. And so forth. In the infinitary languages introduced in earlier publications, the set of conjunctive and disjunctive terms was required to be countable or, more generally, restricted by a fixed cardinal number.

Interest in countable conjunctions and disjunctions in research on infinitary logic is related to the connection between that work and the use of algebraic methods in logic [27]. Any Boolean algebra can be viewed as the set of truth values that can be assigned to propositional atoms. The truth value of a finite formula is calculated by interpreting conjunction, disjunction, and negation as meet, join, and complement. Deductive systems of classical propositional logic are sound with respect to this semantics in the sense that for any Boolean algebra and any assignment of truth values from that algebra to atoms, all provable formulas get the value 1. Algebraic semantics can be extended to propositional formulas with countable conjunctions and disjunctions if the Boolean algebras are required to be σ-complete (every countable subset has a meet and a join). Thus restricting the cardinality of the set of terms in conjunctions and disjunctions is parallel to the completeness conditions studied in the theory of Boolean algebras.

In applications of infinitary formulas to the theory of ASP uncountable conjunctions play an essential role. The formula τE, proposed in [12, Section 4.7] as the representation of an aggregate expression E, is generally an uncountable conjunction.

5.3. Independence of the Generalized Excluded Middle

Replacing the axiom schema (12) by the law of the excluded middle $F \vee \neg F$ would make C^∞ incomplete. This fact is stated without proof by Scott and Tarski [19] (for a Hilbert-style formulation), and it can be proved as follows.

Consider the instance of (12) in which $\{F_\alpha\}_{\alpha \in A}$ is a sequence of distinct atoms:

$$\bigvee_{B \subseteq \mathbf{Z}^+} \left(\bigwedge_{i \in B} p_i \wedge \bigwedge_{i \notin B} \neg p_i \right). \tag{20}$$

Let L be the Boolean algebra of Lebesgue measurable subsets of $[0, 1)$, and let M be the σ-ideal of elements of measure 0. The quotient algebra L/M is complete [28, Corollary 31.1] and consequently provides a semantics for infinitary propositional formulas. The modified system is sound for this semantics. For every positive integer i, let X_i be the set of real numbers x from $[0, 1)$ such that the i-th digit in the binary representation of x is 1. Choose the equivalence class of X_i in the quotient set L/M as the truth value assigned to the atom p_i. Then the truth value of each disjunctive term of (20) is 0. Indeed, for every set B of positive integers, the set

$$\bigcap_{i \in B} X_i \cap \bigcap_{i \notin B} ([0, 1) \setminus X_i)$$

is a singleton (its only element is the number x such that the i-th digit in its binary representation is 1 iff $i \in B$), and singletons are sets of measure 0. Consequently the truth value of (20) is 0 as well. It follows that this formula is not a theorem.

Independence problems of this kind are discussed also by Carol Karp [26].

5.4. Infinitary Deductive Systems

All axioms and inference rules of HT^∞ are included in the "extended system of natural deduction" for infinitary formulas [18]. From the results presented above it is clear that the other axioms of the extended system are redundant. The infinitary De Morgan law

$$\neg \bigwedge_{F \in \mathcal{H}} F \to \bigvee_{F \in \mathcal{H}} \neg F,$$

is one of these redundant axioms. The fact that this formula is a theorem of HT^∞ is proved directly, without a reference to the general completeness theorem, in the preliminary version of this article [14, Section 8].

In the infinitary deductive systems C^∞ and HT^∞ the set of theorems is defined in terms of closure under a set of inference rules; there is no definition of a proof. Accordingly, to justify the claim that a certain formula is a theorem, we need to show that it belongs to every set that includes all axioms and is closed under the inference rules. Such arguments can be represented in a concise form, without mentioning closure under inference rules explicitly [18, Section 3].

It is possible to reformulate the definition of a theorem in an infinitary system in terms of proofs, but such proofs will consist generally of infinitely many formulas, since some inference rules have infinitely many premises.

> In formalized mathematics, proofs are useful in that they are finite syntactic objects that can establish the validity of assertions about infinite domains. "Infinite proofs", on the other hand, do not have this property [29].

The note quoted above defines a correspondence between the validity of infinitary formulas in the logic of here-and-there and the provability of formulas in some finite deductive systems. On the basis of that correspondence, finite proofs can be sometimes used to justify the validity of infinitary formulas.

6. Conclusion

Under the stable model semantics, two sets of propositional formulas are strongly equivalent if and only if they are equivalent in the logic of here-and-there. The proof of this theorem uses equilibrium logic. In this paper, we extended equilibrium logic to infinitary formulas; we defined an infinitary counterpart to the logic of here-and-there and introduced an axiomatization, HT^∞, of that system; finally, we showed that bounded sets of infinitary propositional formulas are strongly equivalent if and only if they are equivalent in HT^∞.

Gebser *et al.* [12, Section 5] used strong equivalence of infinitary formulas to state several useful properties of aggregate expressions. We expect that this concept will find other applications to the theory of answer set programming.

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