Infinitary Equilibrium Logic and Strong Equivalence

Amelia Harrison[1], Vladimir Lifschitz[1], David Pearce[2], and Agustín Valverde[3]

[1] University of Texas, Austin, Texas, USA
ameliaj,vl@cs.utexas.edu
[2] Universidad Politécnica de Madrid, Madrid, Spain
david.pearce@upm.es
[3] University of Malága, Malága, Spain
a_valverde@ctima.uma.es

Abstract. Strong equivalence of logic programs is an important concept in the theory of answer set programming. Equilibrium logic was used to show that propositional formulas are strongly equivalent if and only if they are equivalent in the logic of here-and-there. We extend equilibrium logic to formulas with infinitely long conjunctions and disjunctions, define and axiomatize an infinitary counterpart to the logic of here-and-there, and show that the theorem on strong equivalence holds in the infinitary case as well.

1 Introduction

The original definition of a stable model [3] is restricted to Prolog-style rules—implications with a conjunction of literals in the antecedent and an atom in the consequent. Extending it to arbitrary propositional formulas has been accomplished by two equivalent constructions: using equilibrium logic [13] and using modified reducts [2]. Equilibrium logic served as the basis for the characterization of strong equivalence of logic programs [10] in terms of the logic of Kripke models with two worlds, "the logic of here-and-there." The first axiomatization of that logic was given without proof by Łukasiewicz [11]: add the axiom schema

$$(\neg F \to G) \to (((G \to F) \to G) \to G)) \tag{1}$$

to propositional intuitionistic logic. This axiomatization was rediscovered and proved complete by Thomas [15]. (In the notation of that paper, axiom schema (1) is $3''_2$.) A few years earlier, Umezawa [17] had proposed a simpler axiom schema

$$F \lor (F \to G) \lor \neg G \tag{2}$$

that can be used to axiomatize the logic of here-and-there instead of (1). The completeness of this axiomatization was proved by Hosoi [6].

The definition of a stable model for propositional formulas [2] was extended to formulas with infinitely long conjunctions and disjunctions by Truszczynski [16].

Harrison et al. [4] introduced a deductive system that includes an infinitary counterpart of axiom schema (2) and proved that if two infinitary formulas are equivalent in that system then they are strongly equivalent. Whether the converse holds is posed in that paper as an open question.

In this note, our goals are

(i) to define the infinitary version of the logic of here-and-there,
(ii) to define its nonmonotonic counterpart—the infinitary version of equilibrium logic,
(iii) to verify that stable models of infinitary formulas in the sense of Truszczynski can be characterized in terms of infinitary equilibrium logic,
(iv) to verify that infinitary propositional formulas are strongly equivalent to each other iff they are equivalent in the infinitary logic of here-and-there,
(v) to find an axiomatization of that logic.

The results of this note give a positive answer to the open question mentioned above. Moreover, they show that some axiom schemas introduced by Harrison et al. are redundant.

We will see in Sections 2–5 that achieving goals (i)–(iv) is straightforward, given the work done earlier for finite formulas. Goal (v) is more challenging; see Sections 6, 7. Early work on deductive systems of infinitary propositional formulas [14, 8] was restricted to classical logic. Infinitary intuitionistic logic was studied by Nadel [12]. We are not aware of published work on extending intermediate systems, such as the logic of here-and-there, to infinitary formulas. Additional difficulties arise in connection with the fact that we allow uncountable conjunctions and disjunctions, not covered by Nadel's work.

The main reason why we are interested in stable models of infinitary propositional formulas is that they can be used to define the semantics of the input language of the ASP grounder GRINGO. Consider, for instance, the aggregate expression

```
#count{X:p(X)}==1.
```

Intuitively, it says that the cardinality of the set $\{X \mid p(X)\}$ is 1. If there are infinitely many possible values for X (for instance, if the program uses integers or terms containing function symbols) then this meaning cannot be expressed using a propositional formula. Aggregate expressions like this can be represented by first-order formulas [9], but that method has significant limitations. For example, it is not clear how to apply it to the expression

```
#count{X:p(X)}==Y.
```

Such expressions are included, however, in the subset of the input language of GRINGO studied by Harrison et al [5], who approached the problem of defining the semantics of that language using infinitary propositional formulas. That direction of research shows that the study of strong equivalence of infinitary propositional formulas may be essential for answer set programming.

A preliminary version of this paper was presented at the 2014 Workshop on Answer Set Programming and Other Computing Paradigms.

2 Review: Infinitary Formulas and Their Stable Models

Let σ be a propositional signature, that is, a set of propositional atoms. For every nonnegative integer r, *(infinitary propositional) formulas (over σ) of rank r* are defined recursively, as follows:

- every atom from σ is a formula of rank 0,
- if $\mathcal{H}$ is a set of formulas, and r is the smallest nonnegative integer that is greater than the ranks of all elements of $\mathcal{H}$, then $\mathcal{H}^\wedge$ and $\mathcal{H}^\vee$ are formulas of rank r,
- if F and G are formulas, and r is the smallest nonnegative integer that is greater than the ranks of F and G, then $F \rightarrow G$ is a formula of rank r.

We will write $\{F, G\}^\wedge$ as $F \wedge G$, and $\{F, G\}^\vee$ as $F \vee G$. The symbols $\top$ and $\bot$ will be understood as abbreviations for $\emptyset^\wedge$ and $\emptyset^\vee$ respectively; $\neg F$ stands for $F \rightarrow \bot$, and $F \leftrightarrow G$ stands for $(F \rightarrow G) \wedge (G \rightarrow F)$. These conventions allow us to view finite propositional formulas over σ as a special case of infinitary formulas.

A set or family of formulas is *bounded* if the ranks of its members are bounded from above. For any bounded family $(F_\alpha)_{\alpha \in A}$ of formulas, we denote the formula $\{F_\alpha : \alpha \in A\}^\wedge$ by $\bigwedge_{\alpha \in A} F_\alpha$, and similarly for disjunctions.

Subsets of a signature σ will be also called *interpretations* of σ. The satisfaction relation between an interpretation and a formula is defined recursively, as follows:

- For every atom p from σ, $I \models p$ if $p \in I$.
- $I \models \mathcal{H}^\wedge$ if for every formula F in $\mathcal{H}$, $I \models F$.
- $I \models \mathcal{H}^\vee$ if there is a formula F in $\mathcal{H}$ such that $I \models F$.
- $I \models F \rightarrow G$ if $I \not\models F$ or $I \models G$.

The *reduct* F^I of a formula F w.r.t. an interpretation I is defined recursively, as follows:

- For every atom p from σ, p^I is p if $p \in I$, and $\bot$ otherwise.
- $(\mathcal{H}^\wedge)^I = \{G^I \mid G \in \mathcal{H}\}^\wedge$.
- $(\mathcal{H}^\vee)^I = \{G^I \mid G \in \mathcal{H}\}^\vee$.
- $(G \rightarrow H)^I$ is $G^I \rightarrow H^I$ if $I \models G \rightarrow H$, and $\bot$ otherwise.

An interpretation I is a *stable model* of a set $\mathcal{H}$ of formulas if it is minimal w.r.t. set inclusion among the interpretations satisfying the reducts F^I of all formulas F from $\mathcal{H}$.

3 Infinitary Logic of Here-and-There

An *HT-interpretation* of σ is an ordered pair $\langle I, J \rangle$ of interpretations of σ such that $I \subseteq J$. Intuitively, such a pair describes "two worlds": the atoms in I are true "here" ("in the world H"), and the atoms in J are true "there" ("in the world T").

The satisfaction relation between an HT-interpretation and a formula is defined recursively, as follows:

– For every atom p from σ, $\langle I, J\rangle \models p$ if $p \in I$.
– $\langle I, J\rangle \models \mathcal{H}^\wedge$ if for every formula F in $\mathcal{H}$, $\langle I, J\rangle \models F$.
– $\langle I, J\rangle \models \mathcal{H}^\vee$ if there is a formula F in $\mathcal{H}$ such that $\langle I, J\rangle \models F$.
– $\langle I, J\rangle \models F \rightarrow G$ if
  (i) $\langle I, J\rangle \not\models F$ or $\langle I, J\rangle \models G$, and
  (ii) $J \models F \rightarrow G$.

An *HT-model* of a set $\mathcal{H}$ of infinitary formulas is an HT-interpretation that satisfies all formulas in $\mathcal{H}$.

About a formula F we say that it is *forced in the world* H of an HT-interpretation $\langle I, J\rangle$ if it is satisfied by $\langle I, J\rangle$; we will say that it is *forced in the world* T if it is satisfied by J. The set of worlds in which F is forced will be called the *truth value* of F with respect to $\langle I, J\rangle$. It is easy to check by induction on the rank that every formula that is forced in H is forced in T as well. Consequently, the only possible truth values of a formula are $\emptyset$, $\{T\}$, and $\{H, T\}$.

4 Equilibrium Models

An HT-interpretation $\langle I, J\rangle$ is *total* if $I = J$. It is clear that a total HT-interpretation $\langle J, J\rangle$ satisfies F iff J satisfies F.

An *equilibrium model* of a set $\mathcal{H}$ of infinitary formulas is a total HT-model $\langle J, J\rangle$ of $\mathcal{H}$ such that for every proper subset I of J, $\langle I, J\rangle$ is not an HT-model of $\mathcal{H}$.

The following proposition is similar to Theorem 1 from [2].

Theorem 1 *An interpretation J is a stable model of a set $\mathcal{H}$ of infinitary formulas iff $\langle J, J\rangle$ is an equilibrium model of $\mathcal{H}$.*

Lemma 1. *For any infinitary formula F and any HT-interpretation $\langle I, J\rangle$,*

$$I \models F^J \text{ iff } \langle I, J\rangle \models F.$$

The lemma can be proved by strong induction on the rank of F.

Proof of Theorem 1 It follows from the lemma that a total HT-interpretation $\langle J, J\rangle$ is an equilibrium model of $\mathcal{H}$ iff

– J satisfies all formulas from $\mathcal{H}$, and
– there is no proper subset I of J such that I satisfies the reducts F^J of all formulas F from $\mathcal{H}$.

This condition expresses that J is a stable model of $\mathcal{H}$.

5 Strong Equivalence

About sets $\mathcal{H}_1, \mathcal{H}_2$ of infinitary formulas we say that they are *strongly equivalent* to each other if, for every set $\mathcal{H}$ of infinitary formulas, the sets $\mathcal{H}_1 \cup \mathcal{H}$ and $\mathcal{H}_2 \cup \mathcal{H}$ have the same stable models. About formulas F and G we say that they are *strongly equivalent* if the singleton sets $\{F\}$ and $\{G\}$ are strongly equivalent.

A *unary formula* is an atom or a formula of the form $p \rightarrow q$, where p and q are atoms. The following theorem is similar to the main theorem from [10].

Theorem 2 *For any sets $\mathcal{H}_1, \mathcal{H}_2$ of infinitary formulas, the following conditions are equivalent:*

(i) *$\mathcal{H}_1$ is strongly equivalent to $\mathcal{H}_2$,*
(ii) *for every set $\mathcal{H}$ of unary formulas, sets $\mathcal{H}_1 \cup \mathcal{H}$ and $\mathcal{H}_2 \cup \mathcal{H}$ have the same stable models;*
(iii) *sets $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same HT-models.*

Proof. Clearly, (i) implies (ii). To see that (iii) implies (i), observe that if sets $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same HT-models then $\mathcal{H}_1 \cup \mathcal{H}$ and $\mathcal{H}_2 \cup \mathcal{H}$ have the same HT-models, and consequently have the same equilibrium models. It follows by Theorem 1 that $\mathcal{H}_1 \cup \mathcal{H}$ and $\mathcal{H}_2 \cup \mathcal{H}$ have the same stable models.

It remains to check that (ii) implies (iii). Suppose $\langle I, J\rangle$ is an HT-model of $\mathcal{H}_1$ but not an HT-model of $\mathcal{H}_2$. We will show how to find a set $\mathcal{H}$ of unary formulas such that $\langle J, J\rangle$ is an equilibrium model of one of the sets $\mathcal{H}_1 \cup \mathcal{H}, \mathcal{H}_2 \cup \mathcal{H}$ but not the other. It will follow that the interpretation J is a stable model of one but not the other.

Case 1: $\langle J, J\rangle$ is not an HT-model of $\mathcal{H}_2$. Since $\langle I, J\rangle$ is an HT-model of $\mathcal{H}_1$, it is easy to see that $\langle J, J\rangle$ must be an HT-model of $\mathcal{H}_1$ as well. Then we can take $\mathcal{H} = J$. Indeed, it is clear that $\langle J, J\rangle$ is an HT-model of $\mathcal{H}_1 \cup J$. Furthermore, for any I that is a proper subset of J, $\langle I, J\rangle$ cannot be an HT-model of $\mathcal{H}_1 \cup J$, so that $\langle J, J\rangle$ is an equilibrium model of $\mathcal{H}_1 \cup J$. On the other hand, since $\langle J, J\rangle$ is not a HT-model of $\mathcal{H}_2$, it cannot be an HT-model of $\mathcal{H}_2 \cup J$.

Case 2: $\langle J, J\rangle$ is an HT-model of $\mathcal{H}_2$. Let $\mathcal{H}$ be the set

$$I \cup \{p \rightarrow q \mid p, q \in J \setminus I\}.$$

Since $\langle J, J\rangle$ satisfies every formula in $\mathcal{H}$, it is an HT-model of $\mathcal{H}_2 \cup \mathcal{H}$. To see that it is an equilibrium model, consider any HT-model $\langle K, J\rangle$ of $\mathcal{H}_2 \cup \mathcal{H}$. Clearly, K must contain I. But it cannot be equal to I, since $\langle I, J\rangle$ is not an HT-model of $\mathcal{H}_2$. Thus $I \subset K \subset J$. Consider an atom p in $K \setminus I$ and an atom q in $J \setminus K$. For these atoms, $p \rightarrow q$ belongs to $\mathcal{H}$. But $\langle K, J\rangle$ does not satisfy this implication, contrary to the assumption that it is an HT-model of $\mathcal{H}_2 \cup \mathcal{H}$. We may conclude that $\langle J, J\rangle$ is an equilibrium model of $\mathcal{H}_2 \cup \mathcal{H}$. Finally, we will check that $\langle J, J\rangle$ is not an equilibrium model of $\mathcal{H}_1 \cup \mathcal{H}$. Consider the HT-model $\langle I, J\rangle$ of $\mathcal{H}_1$. Clearly, it is an HT-model of I. Moreover, it satisfies each implication $p \rightarrow q$ in

$\mathcal{H}$: $\langle I, J\rangle$ does not satisfy p because $p \notin I$, and J satisfies q because $q \in J$. We see that $\langle I, J\rangle$ satisfies all formulas in $\mathcal{H}$, so that it is an HT-model of $\mathcal{H}_1 \cup \mathcal{H}$. Furthermore, I is different from J since $\langle J, J\rangle$ is an HT-model of $\mathcal{H}_2$ and $\langle I, J\rangle$ is not. Consequently, I is a proper subset of J, and we may conclude that $\langle J, J\rangle$ is not an equilibrium model of $\mathcal{H}_1 \cup \mathcal{H}$.

A part of any formula can be replaced with a strongly equivalent formula without changing the set of stable models. For instance, it is easy to check that the formulas $p \wedge \neg p$ and $\bot$ are strongly equivalent to each other; it follows that the formulas

$$F \wedge (q \rightarrow (p \wedge \neg p)) \qquad \text{and} \qquad F \wedge \neg q \tag{3}$$

have the same stable models. Corollary 1 expresses a more general fact: several parts (even infinitely many) can be simultaneously replaced by strongly equivalent formulas. Its statement uses the following definitions [4]. Let σ and σ' be disjoint signatures. A *substitution* is a bounded family of formulas over σ with index set σ'. For any substitution ϕ and any formula F over the signature $\sigma \cup \sigma'$, ϕF stands for the formula over σ formed as follows:

- If $F \in \sigma$ then $\phi F = F$.
- If $F \in \sigma'$ then $\phi F = \phi_F$.
- If F is $\mathcal{H}^\wedge$ then $\phi F = \{\phi G \mid G \in \mathcal{H}\}^\wedge$.
- If F is $\mathcal{H}^\vee$ then $\phi F = \{\phi G \mid G \in \mathcal{H}\}^\vee$.
- If F is $G \rightarrow H$ then $\phi F = \phi G \rightarrow \phi H$.

For instance, if $\sigma' = \{r\}$, $\phi_r = p \wedge \neg p$, and $\psi_r = \bot$, then $\phi(F \wedge (q \rightarrow r))$ and $\psi(F \wedge (q \rightarrow r))$ are the formulas (3).

Corollary 1 *Let ϕ and ψ be substitutions such that for all $p \in \sigma'$, ϕ_p is strongly equivalent to ψ_p. Then for any formula F, ϕF is strongly equivalent to ψF, so that ϕF and ψF have the same stable models.*

Proof. By Theorem 2, the assertion of the corollary can be stated as follows: if for all $p \in \sigma'$, ϕ_p and ψ_p are satisfied by the same HT-interpretations, then for any formula F, ϕF and ψF are satisfied by the same HT-interpretations. This is easy to check by induction on the rank of F.

6 An Axiomatization of the Infinitary Logic of Here-and-There

We present an axiomatization HT^∞ of the infinitary logic of here-and-there. The derivable objects in HT^∞ are *(infinitary) sequents*—expressions of the form $\Gamma \Rightarrow F$, where F is an infinitary formula, and Γ is a finite set of infinitary formulas ("F under assumptions Γ"). To simplify notation, we will write Γ as a list. We will identify a sequent of the form $\Rightarrow F$ with the formula F.

The inference rules are the introduction and elimination rules for the propositional connectives

$$(\wedge I)\ \frac{\Gamma \Rightarrow H \quad \text{for all } H \in \mathcal{H}}{\Gamma \Rightarrow \mathcal{H}^\wedge} \qquad (\wedge E)\ \frac{\Gamma \Rightarrow \mathcal{H}^\wedge}{\Gamma \Rightarrow H} \quad (H \in \mathcal{H})$$

$$(\vee I)\ \frac{\Gamma \Rightarrow H}{\Gamma \Rightarrow \mathcal{H}^\vee} \quad (H \in \mathcal{H}) \qquad (\vee E)\ \frac{\Gamma \Rightarrow \mathcal{H}^\vee \qquad \Delta, H \Rightarrow F \quad \text{for all } H \in \mathcal{H}}{\Gamma, \Delta \Rightarrow F}$$

$$(\to I)\ \frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \to G} \qquad (\to E)\ \frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow F \to G}{\Gamma, \Delta \Rightarrow G},$$

where $\mathcal{H}$ is a bounded set of formulas, and the weakening rule

$$(W)\ \frac{\Gamma \Rightarrow F}{\Gamma, \Delta \Rightarrow F}.$$

The set of axioms in HT$^\infty$ is a subset of the set of axioms introduced in the extended system of natural deduction from [4]. HT$^\infty$ includes three axiom schemas:

$$F \Rightarrow F,$$

$$F \vee (F \to G) \vee \neg G, \tag{4}$$

and

$$\bigwedge_{\alpha \in A} \bigvee_{F \in \mathcal{H}_\alpha} F \to \bigvee_{(F_\alpha)_{\alpha \in A}} \bigwedge_{\alpha \in A} F_\alpha \tag{5}$$

for every non-empty family $(\mathcal{H}_\alpha)_{\alpha \in A}$ of sets of formulas such that its union is bounded; the disjunction in the consequent of (5) extends over all elements $(F_\alpha)_{\alpha \in A}$ of the Cartesian product of the family $(\mathcal{H}_\alpha)_{\alpha \in A}$. Axiom schema (4) was mentioned in the introduction in connection with the problem of axiomatizing the logic of here-and-there in the finite case, but now F and G can be infinitary formulas. Axiom schema (5) generalizes (one direction of) the distributivity of conjunction over disjunction to infinitary formulas: if $A = \{1, 2\}$, $\mathcal{H}_1 = \{F_1, G_1\}$, and $\mathcal{H}_2 = \{F_2, G_2\}$, then (5) turns into

$$(F_1 \vee G_1) \wedge (F_2 \vee G_2) \to (F_1 \wedge F_2) \vee (F_1 \wedge G_2) \vee (G_1 \wedge F_2) \vee (G_1 \wedge G_2).$$

The set of *theorems of HT*$^\infty$ is the smallest set of sequents that includes the axioms of the system and is closed under the application of its inference rules. We say that formulas F and G are *equivalent in HT*$^\infty$ if $F \leftrightarrow G$ is a theorem of HT$^\infty$.

The following theorem expresses the soundness and completeness of HT$^\infty$.

Theorem 3 *An infinitary formula F is a theorem of HT$^\infty$ iff it is satisfied by all HT-interpretations.*

The proof of soundness is straightforward. The proof of completeness given in the next section is analogous to the proof of completeness for classical propositional logic from [7].

From Theorems 2 and 3 we conclude:

Corollary 2 *Bounded sets $\mathcal{H}_1$, $\mathcal{H}_2$ of infinitary formulas are strongly equivalent iff $\mathcal{H}_1^\wedge$ is equivalent to $\mathcal{H}_2^\wedge$ in HT$^\infty$.*

7 Proof of Completeness

In the proof of completeness, we use the following construction, due to Cabalar and Ferraris [1, Section 5]. Let $\langle I, J\rangle$ be an HT-interpretation. We define the set M_{IJ} to be

$$I \cup \{\neg\neg p \mid p \in J\} \cup \{\neg p \mid p \in \sigma \setminus J\} \cup \{p \rightarrow q \mid p, q \in J \setminus I\}$$

(recall that σ is the set of all atoms). By $v_{IJ}(F)$ we denote the truth value of F with respect to $\langle I, J\rangle$ (see Section 3). We will omit the subscripts I, J in M_{IJ} and $v_{IJ}(F)$ when it is clear which HT-interpretation we refer to.

Lemma 2. *For any infinitary formula F and HT-interpretation $\langle I, J\rangle$,*

(i) if $v(F) = \emptyset$ then $M^\wedge \Rightarrow \neg F$ is a theorem of HT^∞;
(ii) if $v(F) = \{T\}$ then for every atom q in $J \setminus I$, $M^\wedge \Rightarrow F \leftrightarrow q$ is a theorem of HT^∞;
(iii) if $v(F) = \{H, T\}$ then $M^\wedge \Rightarrow F$ is a theorem of HT^∞.

Proof. We will prove the claim by strong induction on the rank of F. We assume the claim holds for all formulas with rank less than n and show that it holds for a formula F of rank n. We consider cases corresponding to the different possible forms of F and truth values $v(F)$. Note that if $v(F)$ is $\{T\}$ then the set $J \setminus I$ is non-empty. Indeed, if $I = J$ then the truth value of any formula is either $\emptyset$ or $\{H, T\}$.

Case 1: F is an atom.

Case 1.1: $v(F) = \emptyset$. Then $F \in \sigma \setminus J$, and $\neg F \in M$.

Case 1.2: $v(F) = \{T\}$. Then $F \in J \setminus I$, and for every atom q in $J \setminus I$, the implications $F \rightarrow q$ and $q \rightarrow F$ are in M.

Case 1.3: $v(F) = \{H, T\}$. Then $F \in M$.

Case 2: F is of the form $\mathcal{H}^\wedge$. The induction hypothesis is then applicable to all formulas in $\mathcal{H}$.

Case 2.1: $v(F) = \emptyset$. Then there exists a formula G in $\mathcal{H}$ such that $v(G)$ is $\emptyset$. By the induction hypothesis, $M^\wedge \Rightarrow \neg G$ is a theorem of HT^∞. From this we can derive $M^\wedge \Rightarrow \neg(\mathcal{H}^\wedge)$.

Case 2.2: $v(F) = \{T\}$. Let $\mathcal{H}_1$ be the set of all formulas in $\mathcal{H}$ with truth value $\{T\}$, and $\mathcal{H}_2$ be the set of all formulas in $\mathcal{H}$ with truth value $\{H, T\}$. It is clear that $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{H}$ and that $\mathcal{H}_1$ is non-empty. Consider an arbitrary element q of $J \setminus I$. By the induction hypothesis $M^\wedge \Rightarrow G \leftrightarrow q$ is a theorem for every G in $\mathcal{H}_1$, and $M^\wedge \Rightarrow G$ is a theorem for every G in $\mathcal{H}_2$. From these we can derive $M^\wedge \Rightarrow \mathcal{H}_1^\wedge \leftrightarrow q$ and $M^\wedge \Rightarrow \mathcal{H}_2^\wedge$. Then we can derive $M^\wedge \Rightarrow \mathcal{H}^\wedge \leftrightarrow q$.

Case 2.3: $v(F) = \{H, T\}$. Then for each element G in $\mathcal{H}$, $v(G) = \{H, T\}$, and by the induction hypothesis $M^\wedge \Rightarrow G$ is a theorem. From these sequents we can derive $M^\wedge \Rightarrow \mathcal{H}^\wedge$.

Case 3: F is of the form $\mathcal{H}^\vee$. The induction hypothesis is then applicable to all formulas in $\mathcal{H}$.

Case 3.1: $v(F) = \emptyset$. Then for each element G in $\mathcal{H}$, $v(G) = \emptyset$, and by the induction hypothesis $M^\wedge \Rightarrow \neg G$ is a theorem. From these sequents we can derive $M^\wedge \Rightarrow \neg(\mathcal{H}^\vee)$.

Case 3.2: $v(F) = \{T\}$. Let $\mathcal{H}_1$ be the set of all formulas in $\mathcal{H}$ with truth value $\{T\}$, and $\mathcal{H}_2$ be the set of all formulas in $\mathcal{H}$ with truth value $\emptyset$. It is clear that $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{H}$ and that $\mathcal{H}_1$ is non-empty. Consider an arbitrary element q of $J \setminus I$. By the induction hypothesis $M^\wedge \Rightarrow G \leftrightarrow q$ is a theorem for every G in $\mathcal{H}_1$, and $M^\wedge \Rightarrow \neg G$ is a theorem for every G in $\mathcal{H}_2$. From these we can derive $M^\wedge \Rightarrow \mathcal{H}_1^\vee \leftrightarrow q$ and $M^\wedge \Rightarrow \neg(\mathcal{H}_2^\vee)$. Then we can derive $M^\wedge \Rightarrow \mathcal{H}^\vee \leftrightarrow q$.

Case 3.3: $v(F) = \{H, T\}$. Then there exists a formula G in $\mathcal{H}$ such that $v(G)$ is $\{H, T\}$. By the induction hypothesis, $M^\wedge \Rightarrow G$ is a theorem. From this we can derive $M^\wedge \Rightarrow \mathcal{H}^\vee$.

Case 4: F is of the form $F_1 \rightarrow F_2$. The induction hypothesis is then applicable to F_1 and F_2.

Case 4.1: $v(F) = \emptyset$. Then $v(F_1)$ is non-empty and $v(F_2)$ is empty.

Case 4.1.1: $v(F_1) = \{T\}$. By the induction hypothesis $M^\wedge \Rightarrow \neg F_2$ is a theorem, as is $M^\wedge \Rightarrow F_1 \leftrightarrow q$ for any q in $J \setminus I$. Consider an atom q in $J \setminus I$. By the construction of M, we know that $\neg\neg q$ is an element of M. From the sequents $M^\wedge \Rightarrow F_1 \leftrightarrow q$, $M^\wedge \Rightarrow \neg F_2$, and $M^\wedge \Rightarrow \neg\neg q$, we can derive $M^\wedge \Rightarrow \neg(F_1 \rightarrow F_2)$.

Case 4.1.2: $v(F_1) = \{H, T\}$. By the induction hypothesis, both $M^\wedge \Rightarrow F_1$ and $M^\wedge \Rightarrow \neg F_2$ are theorems. From these sequents we can derive $M^\wedge \Rightarrow \neg(F_1 \rightarrow F_2)$.

Case 4.2: $v(F) = \{T\}$. Then $v(F_1) = \{H, T\}$ and $v(F_2) = \{T\}$. By the induction hypothesis $M^\wedge \Rightarrow F_2 \leftrightarrow q$ is a theorem for any $q \in J \setminus I$, and $M^\wedge \Rightarrow F_1$ is a theorem as well. From these two sequents we can derive $M^\wedge \Rightarrow (F_1 \rightarrow F_2) \leftrightarrow q$.

Case 4.3: $v(F) = \{H, T\}$.

Case 4.3.1: $v(F_1) = \emptyset$. Then by the induction hypothesis $M^\wedge \Rightarrow \neg F_1$ is a theorem. From this we can derive $M^\wedge \Rightarrow F_1 \rightarrow F_2$.

Case 4.3.2: $v(F_2) = \{H, T\}$. Then by the induction hypothesis $M^\wedge \Rightarrow F_2$ is a theorem. From this we can derive $M^\wedge \Rightarrow F_1 \rightarrow F_2$.

Case 4.3.3: $v(F_1) \neq \emptyset$ and $v(F_2) \neq \{H, T\}$. Since $v(F)$ is $\{H, T\}$, $v(F_1)$ is different from $\{H, T\}$ and therefore must be equal to $\{T\}$. It follows that $v(F_2)$ is different from $\emptyset$, and therefore must be $\{T\}$ also. Consider an element q in $J \setminus I$. By the induction hypothesis both $M^\wedge \Rightarrow F_1 \leftrightarrow q$ and $M^\wedge \Rightarrow F_2 \leftrightarrow q$ are theorems. From these two sequents we can derive $M^\wedge \Rightarrow F_1 \rightarrow F_2$.

Note that in the proof of the lemma we did not refer to axiom schemas (4) and (5); the assertion of the lemma would hold even if those axioms were removed from HT^∞.

Lemma 3. *The disjunction of the formulas $M^{\wedge}_{IJ}$ over all HT-interpretations $\langle I, J\rangle$ is a theorem of HT^{∞}.*

Proof. Let Q stand for the set of disjunctions

$$p \vee (p \to q) \vee \neg q, \tag{6}$$

$$\neg p \vee \neg\neg p \tag{7}$$

for all p, q from σ. Let $(\mathcal{H}_D)_{D\in Q}$ be the following family of sets:

$$\begin{array}{ll} \mathcal{H}_D = \{p, p \to q, \neg q\} & \text{if } D = p \vee (p \to q) \vee \neg q; \\ \mathcal{H}_D = \{\neg p, \neg\neg p\} & \text{if } D = \neg p \vee \neg\neg p. \end{array}$$

Then the formula

$$\bigwedge_{D\in Q} \bigvee_{S\in\mathcal{H}_D} S \to \bigvee_{(S_D)_{D\in Q}} \bigwedge_{D\in Q} S_D,$$

(where the disjunction in the consequent extends over all elements $(S_D)_{D\in Q}$ of the Cartesian product of the family $(\mathcal{H}_D)_{D\in Q}$) is an instance of axiom schema (5). Since the antecedent of this implication is the conjunction of all formulas in Q, it is a theorem of HT^{∞}. It follows that the consequent is a theorem as well. To complete the proof it is sufficient to show that for every disjunctive term

$$\bigwedge_{D\in Q} S_D \tag{8}$$

of the consequent there exists an HT-interpretation $\langle I, J\rangle$ such that the sequent

$$\bigwedge_{D\in Q} S_D \Rightarrow M^{\wedge}_{IJ} \tag{9}$$

is a theorem.

Consider one of the conjunctions (8), and let C be set of its conjunctive terms. The elements of C are formulas of the forms

$$p,\ \neg p,\ \neg\neg p,\ p \to q.$$

If C contains both a formula and its negation then (9) is a theorem for every $\langle I, J\rangle$. Otherwise, let I denote the set of all atoms in C, and J denote the set of all atoms p such that $\neg\neg p$ is in C. Let us check that $I \subseteq J$. Assume $p \in I$ so that $p \in C$. Since C is consistent, it does not contain $\neg p$, and since it contains a term from each disjunction (7), it contains $\neg\neg p$. So $\langle I, J\rangle$ is an HT-interpretation.

We will show that every formula from M_{IJ} belongs to C. By the choice of I, $I \subseteq C$. By the choice of J, $\{\neg\neg p \mid p \in J\} \subseteq C$. Consequently $\{\neg p \mid p \in \sigma\setminus J\} \subseteq C$, because C contains one term from each disjunction (7). Finally, we need to check that $\{p \to q \mid p, q \in J \setminus I\} \subseteq C$. Consider a pair of atoms p, q that occur in J but not in I. By the choice of I, p is not in C, and by the choice of J, $\neg q$ is not in C. Since C contains one term from each of the disjunctions (6) and contains neither p nor $\neg q$, C must contain $p \rightarrow q$.

Proof of Completeness Let F be an infinitary formula over signature σ that is satisfied by all HT-interpretations of σ. By Lemma 2(iii), $M_{IJ} \Rightarrow F$ is a theorem of HT$^\infty$ for all HT-interpretations $\langle I, J\rangle$. By Lemma 3, it follows that F is a theorem also.

It is clear from the proof that HT$^\infty$ will remain complete if we require that formulas F and G in axiom schema (4) must be literals, and that the sets $\mathcal{H}_i$ in axiom schema (5) must be finite.

8 Example: Infinitary De Morgan's Law

As observed in Section 6, the set of axioms in HT$^\infty$ is a subset of the set of axioms introduced in the extended system of natural deduction from [4]. From the results presented in this note it is clear that the other axioms in the extended system are redundant. The infinitary De Morgan's law,

$$\neg \bigwedge_{F\in\mathcal{H}} F \rightarrow \bigvee_{F\in\mathcal{H}} \neg F, \tag{10}$$

is one of these redundant axioms. In this section, we show directly, without a reference to the general completeness theorem, how to prove (10) in HT$^\infty$.

Let Q stand for the set of disjunctions

$$F \vee (F \rightarrow G) \vee \neg G, \tag{11}$$

for all formulas F, G from $\mathcal{H}$. Let $(\mathcal{H}_D)_{D\in Q}$ be the following family of sets:

$$\mathcal{H}_D = \{F, F \rightarrow G, \neg G\}.$$

Then the formula

$$\bigwedge_{D\in Q} \bigvee_{S\in\mathcal{H}_D} S \rightarrow \bigvee_{(S_D)_{D\in Q}} \bigwedge_{D\in Q} S_D, \tag{12}$$

(where the disjunction in the consequent extends over all elements $(S_D)_{D\in Q}$ of the Cartesian product of the family $(\mathcal{H}_D)_{D\in Q}$) is an instance of axiom schema (5). Since the antecedent of this implication is the conjunction of all formulas in Q, it is a theorem of HT$^\infty$. It follows that the consequent is a theorem as well. To complete the proof it is sufficient to show that from the antecedent of (10) and any disjunctive term

$$\bigwedge_{D\in Q} S_D \tag{13}$$

of the consequent of (12), we can derive the consequent of (10). Consider one of the conjunctions (13), and let C be set of its conjunctive terms. The elements of C are formulas of the forms

$$F,\ F \rightarrow G,\ \neg G.$$

If C contains $\neg F$ for some formula F then the consequent of (10) follows immediately. Otherwise, we will show that assuming $C^{\wedge}$ and any element F of $\mathcal{H}$ we can derive

$$\bigwedge_{F \in \mathcal{H}} F, \tag{14}$$

contradicting the antecedent of (10), and allowing us to derive $\neg F$ from $C^{\wedge}$ and the antecedent of (10). If C contains every formula F in $\mathcal{H}$ then (14) is immediate. Otherwise, there is some G from $\mathcal{H}$ which is not in C. Assume G. Since G is not in C and C does not contain the negation of any formula, we may conclude that C contains $G \rightarrow F$ for all formulas F from $\mathcal{H}$. It follows that from G and $C^{\wedge}$ we can derive (14).

9 Conclusion

Under the stable model semantics, two sets of propositional formulas are strongly equivalent if and only if they are equivalent in the logic of here-and-there. This theorem was originally proved using equilibrium logic in [10]. In this paper, we extended equilibrium logic to infinitary formulas; we defined an infinitary counterpart to the logic of here-and-there and introduced an axiomatization, HT^{∞}, of that system; finally, we showed that bounded sets of infinitary propositional formulas are strongly equivalent if and only if they are equivalent in HT^{∞}.

Acknowledgements

Thanks to Pedro Cabalar and Yuliya Lierler, and to the anonymous referees for helpful comments. The first two authors were partially supported by the National Science Foundation under Grant IIS-1422455.

References

1. Cabalar, P., Ferraris, P.: Propositional theories are strongly equivalent to logic programs[4]. Theory and Practice of Logic Programming 7 (2007)
2. Ferraris, P.: Answer sets for propositional theories. In: Proceedings of International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR). pp. 119–131 (2005)
3. Gelfond, M., Lifschitz, V.: The stable model semantics for logic programming. In: Kowalski, R., Bowen, K. (eds.) Proceedings of International Logic Programming Conference and Symposium. pp. 1070–1080. MIT Press (1988)
4. Harrison, A., Lifschitz, V., Truszczynski, M.: On equivalence of infinitary formulas under the stable model semantics. Theory and Practice of Logic Programming 15 (2015)
5. Harrison, A., Lifschitz, V., Yang, F.: The semantics of Gringo and infinitary propositional formulas. In: Proceedings of International Conference on Principles of Knowledge Representation and Reasoning (KR) (2014)

[4] http://www.cs.utexas.edu/users/otto/papers.html

6. Hosoi, T.: The axiomatization of the intermediate propositional systems S_n of Gödel. Journal of the Faculty of Science of the University of Tokyo 13, 183–187 (1966)
7. Kalmár, L.: Zurückführung des Entscheidungsproblems auf den Fall von Formeln mit einer einzigen, bindren, Funktionsvariablen. Compositio Mathematica 4, 137–144 (1936)
8. Karp, C.R.: Languages with expressions of infinite length. North-Holland, Amsterdam (1964)
9. Lee, J., Lifschitz, V., Palla, R.: A reductive semantics for counting and choice in answer set programming. In: Proceedings of the AAAI Conference on Artificial Intelligence (AAAI). pp. 472–479 (2008)
10. Lifschitz, V., Pearce, D., Valverde, A.: Strongly equivalent logic programs. ACM Transactions on Computational Logic 2, 526–541 (2001)
11. Łukasiewicz, J.: Die Logik und das Grundlagenproblem. In: Les Entretiens de Zürich sue les Fondements et la méthode des sciences mathématiques 1938, pp. 82–100. Leemann, Zürich (1941)
12. Nadel, M.: Infinitary intuitionistic logic from a classical point of view. Annals of Mathematical Logic pp. 159–191 (1978)
13. Pearce, D.: A new logical characterization of stable models and answer sets. In: Dix, J., Pereira, L., Przymusinski, T. (eds.) Non-Monotonic Extensions of Logic Programming (Lecture Notes in Artificial Intelligence 1216). pp. 57–70. Springer (1997)
14. Scott, D., Tarski, A.: The sentential calculus with infinitely long expressions. In: Colloquium Mathematicae. vol. 6, pp. 165–170 (1958)
15. Thomas, I.: Finite limitations on dummett's LC. Notre Dame Journal of Formal Logic III(3) (July 1962)
16. Truszczynski, M.: Connecting first-order ASP and the logic FO(ID) through reducts. In: Correct Reasoning: Essays on Logic-Based AI in Honor of Vladimir Lifschitz. Springer (2012)
17. Umezawa, T.: On intermediate many-valued logics. Journal of the Mathematical Society of Japan 11(2) (1959)