A Normal Form for Rules Containing Arithmetic Operations

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Abstract

This paper describes the process of translating rules that may contain arithmetic operations into the language of first-order logic. It identifies a normal form for which this transformation can be performed in a particularly simple and natural way. Other rules can be converted to this normal form by steps that preserve their meaning under the stable model semantics.

1 Introduction

This paper is about translating rules in the sense of answer set programming (ASP) (Marek and Truszczynski 1999; Niemelä 1999; Lifschitz 2019) into the language of first-order logic. Such translations help us clarify the relationship between logic programs and first-order theories as knowledge representation mechanisms. They are important also because of their role in the design of ANTHEM (Fandinno et al. 2025), a proof assistant for verifying ASP programs that operates by transforming verification tasks into first-order reasoning problems. We are interested in translations that are sound with respect to the stable model semantics, which has been defined both for ASP rules (Gebser et al. 2015) and for first-order formulas (Pearce and Valverde 2005; Ferraris, Lee, and Lifschitz 2011; Truszczynski 2012).

To illustrate some of the issues involved, compare the rule

$$q(X, Y+1) \leftarrow p(X, Y) \tag{1}$$

with its counterpart in the language of first-order logic:

$$\forall XY(p(X, Y) \rightarrow q(X, Y+1)).$$

When rule (1) is grounded, substituting symbolic constants for Y is not allowed, because Y occurs in the scope of an arithmetic operation (Calimeri et al. 2020, Section 3). To reflect this difference between X and Y in the formula above, we have to distinguish between variables of the sort *general*, such as X, and variables of the subsort *integer*, such as Y in this example. The syntax of the language requires that all arguments of an arithmetic operation be integer terms.

Consider now a rule containing the integer division operation:

$$q(X/Y+1) \leftarrow p(X, Y). \tag{2}$$

Since division is not a total function on domains that include zero, the expression X/Y with integer variables X, Y cannot be allowed in a first-order language: the semantics of these languages requires that every function symbol represent a total function. Transforming rule (2) into a formula has to be less straightforward and less natural than in the case of rule (1).

Translating rules containing placeholders (Gebser et al. 2019, Section 3.15) may be challenging in another way. Placeholders for integers are symbolic constants that are expected to be assigned numeric values before executing the program, and they may occur in the scope of an arithmetic operation. The fact $p(n+1)$, for instance is not a syntactically correct formula in the two-sorted language outlined above, because the first argument of addition in this expression is not an integer term. This fact can be represented by the longer formula

$$\forall I(I = n \rightarrow p(I+1)), \tag{3}$$

where I is an integer variable.

ANTHEM 2.0 implements two procedures for converting rules into formulas, τ^* and ν (Fandinno et al. 2025, Section 3.1). Each of them applies to rules in a fragment of the input language of the ASP grounder GRINGO (Gebser et al. 2019, Section 3.1), called mini-GRINGO, and generates formulas in a two-sorted first-order language. The translation τ^* (Lifschitz, Lühne, and Schaub 2019, Section 6) is applicable to all mini-GRINGO rules, whereas ν (Lifschitz 2021) is limited to rules satisfying some syntactic conditions. For example, ν is not applicable to rule (2). On the other hand, when ν is applicable, the result is often simpler and more natural than the result of applying τ^*. This makes ν more attractive for the user of ANTHEM, who often needs to read formulas representing rules.

In this paper, we propose a method of representing rules by formulas that combines the advantages of τ^* and ν: on the one hand, it is applicable to all mini-GRINGO rules; on the other, it often produces relatively simple translations. The process involves converting the given mini-GRINGO rule to a certain normal form. For example, the normal form of rule (2) is

$$q(Z+1) \leftarrow p(X, Y) \wedge Z = X/Y. \tag{4}$$

The normal form of the fact $p(n+1)$ is

$$p(X+1) \leftarrow X = n. \tag{5}$$

The normal form of the fact $p(1\,..\,8)$ ("p holds for all numbers between 1 and 8") is

$$p(X) \leftarrow X = 1\,..\,8. \tag{6}$$

Rules in normal form can be transformed into two-sorted formulas using a generalization of the translation ν. For example, the result of applying generalized ν to rule (5) is formula (3).

The role of the normal form described in this paper is determined by two theorems. First, if N is the normal form of a mini-GRINGO rule R then $\tau^*(N)$ is intuitionistically equivalent to $\tau^*(R)$. The fact that conversion to normal form can be justified without using the law of the excluded middle is essential because intuitionistically acceptable transformations are known to preserve stable models of a first-order theory (Pearce and Valverde 2005). Full classical logic does not have this property. For instance, if a first-order theory includes the axiom $\forall X(\neg p(X) \to q(X))$ then replacing that axiom by the classically equivalent formula $\forall X(\neg q(X) \to p(X))$ may change its stable models.

Second, the result of applying the generalized ν to any rule N in normal form is inuitionistically equivalent to $\tau^*(N)$.

In Sections 2 and 3 we review the syntax of mini-GRINGO rules and the definition of the translation τ^*. Our version of mini-GRINGO is more expressive than the class of programs studied in earlier work on transforming rules into formulas, because we allow symbolic constants to be used as function symbols. It is also more general than earlier work in the sense that we deal here with arbitrary dialects of mini-GRINGO (Lifschitz 2025), instead of one specific dialect. The process of converting rules to normal form is described in Section 4, and the generalized natural translation in Section 5. In Section 6, we show how formulas produced by the natural translation can be made completable. This additional transformation is required for constructing the completion of a mini-gringo program (Fandinno, Lifschitz, and Temple 2024, Section 3.1)—a step that plays an important role in the operation of ANTHEM (Fandinno et al. 2025, Section 3.2.2). Proofs of the main results of this paper are outlined in Sections 7 and 8.

2 Review: the syntax of mini-GRINGO

The syntax of rules is determined by a *mini-*GRINGO *signature*, which consists of

- a set of *basic symbols*, with some of them designated as *symbolic constants*,
- a countably infinite set of *variables*, disjoint from the set of basic symbols, and
- a set of *arithmetic function symbols*, with a positive integer, called its *arity*, assigned to each of them.

For example, we can define a mini-GRINGO signature Σ_0 by saying that symbolic constants are strings of letters and digits that begin with a lowercase letter; basic symbols are symbolic constants and integers (both positive and negative) in decimal notation; variables are strings of letters and digits that begin with an uppercase letter; the binary arithmetic function symbols are

$$.. \quad + \quad - \quad \times \quad /$$

*(Mini-*GRINGO*) terms* over a mini-GRINGO signature Σ are defined recursively:

- all basic symbols and variables of Σ are terms;
- if c is a symbolic constant of Σ, and $\mathbf{t}$ is a non-empty tuple of terms (separated by commas), then $c(\mathbf{t})$ is a term;
- if f is a k-ary arithmetic function symbol of Σ, and $\mathbf{t}$ is a k-tuple of terms, then $f(\mathbf{t})$ is a term.

A term is *precomputed* if it contains neither variables nor arithmetic function symbols. In other words, precomputed terms are formed from basic symbols using symbolic function symbols.

Atoms over Σ are symbolic constants and expressions of the form $c(\mathbf{t})$, where $\mathbf{t}$ is a non-empty tuple of terms. The expression $c()$ is alternative notation for the atom c. A *literal* over Σ is an atom possibly preceded by one or two occurrences of *not*. *Comparisons* over Σ are expressions of the forms $t_1 \prec t_2$ where t_1 and t_2 are terms and $\prec$ is one of the binary relation symbols

$$= \quad \neq \quad < \quad \leq \quad > \quad \geq$$

A *rule* over Σ is an expression of the form

$$H \leftarrow B_1 \wedge \cdots \wedge B_n \tag{7}$$

($n \geq 0$), where the head H is either an atom or empty, and each member B_i of the body is a literal or a comparison. Rules with the empty body are *facts*, and rules with the empty head are *constraints*. Rules of the form

$$H \leftarrow B_1 \wedge \cdots \wedge B_n \wedge \textit{not not } H,$$

where H is an atom, are called *choice rules* and can be written as

$$\{H\} \leftarrow B_1 \wedge \cdots \wedge B_n.$$

A *dialect* over a mini-GRINGO signature consists of

- a set of basic symbols called *numerals*, disjoint from the set of symbolic constants,
- a 1–1 correspondence $n \mapsto \overline{n}$ between the set $\mathbb{Z}$ of integers and the set of numerals,
- a total order on precomputed terms such that numerals are contiguous (every term between two numerals is a numeral), and, for all integers m and n, $\overline{m} \leq \overline{n}$ iff $m \leq n$,
- for every arithmetic function symbol f, a function $\hat{f}$ that maps $\mathbb{Z}^k$ to the powerset of $\mathbb{Z}$, where k is the arity of f.[1]

For example, we can define a dialect D_0 of mini-GRINGO over the signature Σ_0 by saying that numerals are integers written in decimal notation; that $\overline{n}$ is the representation of n in decimal notation; that basic symbols are ordered as in the language ASP-Core-2 (Calimeri et al. 2020, Section 3); and that the functions $\widehat{..}$, $\widehat{+}$, $\widehat{-}$ and $\widehat{\times}$ are defined by the formulas

$$\widehat{..}(m,n) = \{k : m \leq k \leq n\};$$
$$\widehat{+}(m,n) = \{m+n\}; \qquad \widehat{-}(m,n) = \{m-n\};$$
$$\widehat{\times}(m,n) = \{m \times n\};$$
$$\text{if } n \neq 0 \text{ then } \widehat{/}(m,n) = \{\lfloor m/n \rfloor\}; \quad \widehat{/}(m,0) = \emptyset.$$

[1] This description of $\hat{f}$ is different than in the paper by Lifschitz (2025). That version would not allow us to include the interval symbol .. among arithmetic function symbols.

In this example, every basic symbol is either a symbolic constant or a numeral. The signature Σ_0 can be extended by adding `#sup` and `#inf` (Gebser et al. 2019, Section 3.1.1) to the set of basic symbols; those would be neither symbolic constants nor numerals.

3 Review: translation τ^*

3.1 Target language of τ^*

An arithmetic function symbol f of a dialect of mini-GRINGO is *definite* if all values of $\hat{f}$ are singletons, and *indefinite* otherwise. In the dialect D_0, the symbols .. and $/$ are indefinite (the latter because $\widehat{/}(m, 0)$ is empty). This distinction is important because representing definite function symbols in a first-order language is particularly easy.

For any dialect D of mini-GRINGO, by $\sigma(D)$ we denote the two-sorted signature that[2] consists of

- the sort *general* and its subsort *integer*;
- all numerals of D as object constants of sort *integer*;
- all basic symbols of D other than numerals as object constants of sort *general*;
- expressions $c\backslash k$, where c is a symbolic constant of D and k is a positive integer, as k-ary function constants, with both arguments and value of sort *general*;
- all definite arithmetic function symbols of D as function constants with both arguments and value of sort *integer*;
- expressions c/k, where c is a symbolic constant of D and k is a nonnegative integer, as k-ary predicate constants with arguments of sort *general*;
- the symbol $\leq$ as a binary predicate constant with both arguments of sort *general*.

For example, the signature $\sigma(D_0)$ includes the function symbols $+$, $-$ and $\times$, but neither the division symbol nor the interval symbol.

Order relations other than $\leq$ are defined as abbreviations.

We identify variables of the sort *general* with variables of the dialect D.

3.2 Definable symbols

By $\sigma^-(D)$ we denote the signature obtained from $\sigma(D)$ by removing all the predicate constants c/k. The *standard interpretation* S of $\sigma^-(D)$ is defined as follows:

- its domain of the sort *general* is the set of all precomputed terms of D,
- its domain of the sort *integer* is the set of all numerals of D,
- if r is a basic symbol of D then $r^I = r$;
- for every function constant $c\backslash k$ and precomputed terms $t_1, \dots, t_k$ of D,

$$(c\backslash k)^I(t_1, \dots, t_k) = c(t_1, \dots, t_k);$$

- for every definite arithmetic function symbol f of D and integers $n_1, \dots, n_k$, where k is the arity of f, $f^I(\overline{n_1}, \dots, \overline{n_k}) = \overline{n}$ for the integer n such that $\hat{f}(n_1, \dots, n_k) = \{n\}$;
- $\leq^I$ is the order relation of the dialect D.

The set of sentences over $\sigma^-(D)$ that are satisfied by the standard interpretation S is denoted by $Std(D)$.

Let f be a k-ary arithmetic function symbol of D, and let $F(I_1, \dots, I_{k+1})$ be a formula over the signature $\sigma^-(D)$ that contains neither symbolic object constants nor free variables other than the integer variables $I_1, \dots, I_{k+1}$. We say that the formula $F(I_1, \dots, I_{k+1})$ *defines* f if, for all integers $n_1, \dots, n_{k+1}$,

$$n_{k+1} \in \hat{f}(n_1, \dots, n_k) \text{ iff } F(\overline{n_1}, \dots, \overline{n_{k+1}}) \in Std(D). \quad (8)$$

If such a formula exists then we say that the symbol f is *definable*.

Every definite function symbol f is definable: we can take $F(I_1, \dots, I_{k+1})$ to be $f(I_1, \dots, I_k) = I_{k+1}$. Indeed, the formula $f(\overline{n_1}, \dots, \overline{n_k}) = \overline{n_{k+1}}$ is satisfied by the standard interpretation iff $\hat{f}(n_1, \dots, n_k) = \{n_{k+1}\}$, which is equivalent to the left-hand side of (8).

In the dialect D_0, all arithmetic function symbols are definable. Indeed, $+$, $-$ and $\times$ are definable because they are definite. The formula $I_1 \leq I_3 \leq I_2$ defines the interval symbol, and the formula

$$(0 \leq I_1 - I_2 \times I_3 < I_2) \vee (0 \geq I_1 - I_2 \times I_3 > I_2)$$

defines integer division.

In the rest of the paper, D is a dialect of mini-GRINGO with at least one symbolic constant, such that all its arithmetic function symbols are definable.

3.3 Values of a term

For a mini-GRINGO term t and a general variable V that does not occur in t, the formula $val_t(V)$ over $\sigma(D)$, defined below, expresses that V is a value of t. In writting this definition, it is convenient to use the following notation: If $\mathbf{t}$ and $\mathbf{V}$ are lists of terms $t_1, \dots, t_k$ and variables $V_1, \dots, V_k$ of the same length, respectively, then $val_{\mathbf{t}}(\mathbf{V})$ stands for the conjunction $val_{t_1}(V_1) \wedge \dots \wedge val_{t_k}(V_k)$. The definition is recursive:

- if t is a basic symbol or a variable then $val_t(V)$ is $V = t$;
- if t is $c(\mathbf{t})$, where c is a symbolic constant, then $val_t(V)$ is

$$\exists \mathbf{Y}(val_{\mathbf{t}}(\mathbf{Y}) \wedge V = (c\backslash k)(\mathbf{Y})),$$

  where $\mathbf{Y} = Y_1, \dots, Y_k$ are general variables that do not occur in t;
- if t is $f(\mathbf{t})$, where f is a definite arithmetic function symbol, then $val_t(V)$ is

$$\exists \mathbf{I}(val_{\mathbf{t}}(\mathbf{I}) \wedge V = f(\mathbf{I})),$$

  where $\mathbf{I} = I_1, \dots, I_k$ are integer variables;

[2] Syntax and semantics of many-sorted languages are reviewed by Fandinno, Lifschitz, and Temple (2024) in Appendix A.

$$\forall X_1 X_2 XY(X_1 = X \wedge \exists I_1 I_2(I_1 = Y \wedge I_2 = 1 \wedge X_2 = I_1 + I_2) \wedge (p/2)(X_1, X_2) \rightarrow (q/2)(X_1, X_2)), \quad (9)$$

$$\forall X_1(\exists I_1 I_2 I_3(I_1 = 1 \wedge I_2 = 8 \wedge I_3 = X_1 \wedge I_1 \leq I_3 \leq I_2) \rightarrow (p/1)(X_1)) \quad (10)$$

Figure 1: Result of applying τ^* to rule (1) and fact $p(1..8)$

- if t is $f(\mathbf{t})$, where f is an indefinite arithmetic function symbol, then $val_t(V)$ is

$$\exists \mathbf{I} I_{k+1}(val_{\mathbf{t}}(\mathbf{I}) \wedge I_{k+1} = V \wedge F(\mathbf{I}, I_{k+1})),$$

where $\mathbf{I} = I_1, \ldots, I_k$ and I_{k+1} are integer variables, and $F(I_1, \ldots, I_{k+1})$ is a formula that defines f.

For example, the formula $val_X(V)$ is $V = X$. In the dialect D_0, $val_{Y+1}(V)$ is

$$\exists I_1 I_2(I_1 = Y \wedge I_2 = 1 \wedge V = I_1 + I_2),$$

and $val_{1\,..\,8}(V)$ is

$$\exists I_1 I_2 I_3(I_1 = 1 \wedge I_2 = 8 \wedge I_3 = V \wedge I_1 \leq I_3 \leq I_2).$$

3.4 Definition of τ^*

The translation τ^B, defined below, produces a formula that characterizes the meaning of an expression in the body of a mini-GRINGO rule. It transforms

- $c(\mathbf{t})$, where c is a symbolic constant, into

$$\exists \mathbf{X}(val_{\mathbf{t}}(\mathbf{X}) \wedge (c/k)(\mathbf{X}));$$

- *not* $c(\mathbf{t})$ into

$$\exists \mathbf{X}(val_{\mathbf{t}}(\mathbf{X}) \wedge \neg(c/k)(\mathbf{X}));$$

- *not not* $c(\mathbf{t})$ into

$$\exists \mathbf{X}(val_{\mathbf{t}}(\mathbf{X}) \wedge \neg\neg(c/k)(\mathbf{X}));$$

- $t_1 \prec t_2$, into $\exists X_1 X_2(val_{t_1}(X_1) \wedge val_{t_2}(X_2) \wedge X_1 \prec X_2)$;

where $\mathbf{X} = X_1, \ldots, X_k$ and $\mathbf{t} = t_1, \ldots, t_k$ respectively are lists of general variables and terms of the same length such that no X_i occurs in $\mathbf{t}$.

For example, the result of applying τ^B to the atom $p(X, Y)$ is

$$\exists X_1 X_2(X_1 = X \wedge X_2 = Y \wedge (p/2)(X_1, X_2)).$$

If *Body* is a conjunction $B_1 \wedge \cdots \wedge B_n$ of literals and comparisons then $\tau^B(\mathit{Body})$ stands for the conjunction $\tau^B(B_1) \wedge \cdots \wedge \tau^B(B_n)$.

Now we are ready to define the translation τ^*. It converts a rule

$$c(\mathbf{t}) \leftarrow \mathit{Body}$$

into the formula

$$\widetilde{\forall}(val_{\mathbf{t}}(\mathbf{X}) \wedge \tau^B(\mathit{Body}) \rightarrow (c/k)(\mathbf{X})),$$

where $\mathbf{X} = X_1, \ldots, X_k$ and $\mathbf{t} = t_1, \ldots, t_k$ respectively are lists of general variables and terms of the same length, and $\widetilde{\forall}$ denotes universal closure; and a constraint $\leftarrow$ *Body* into $\widetilde{\forall} \neg \tau^B(\mathit{Body})$.

For example, the result of applying τ^* to rule (1) is the sentence (9) in Figure 1, and the result of applying τ^* to fact $p(1\,..\,8)$ is the sentence (10) in the same figure.

4 Normal form

An *equation* is a comparison of the form $t_1 = t_2$. An equation is *numeric* if all basic symbols occurring in it are numerals.

A rule is *in normal form* if

(a) every basic symbol that occurs in it in the scope of an arithmetic function symbol is a numeral, and

(b) every occurrence of a term of the form $f(\mathbf{t})$ with indefinite f is the right-hand side of a numeric equation.

For example, all rules that do not contain arithmetic function symbols are in normal form. In the dialect D_0, rules (1) and (4)–(6) are in normal form. Rule (2) and the fact $p(1\,..\,8)$ are not in normal form because they violate condition (b). The facts $p(n+1)$ and $p(n(3)+1)$ violate condition (a).

We will show that any rule can be converted to normal form by

replacing an occurrence of some term t by a fresh variable V and adding the equation $V = t$ to the body of the rule (11)

one or more times.

The description of this process uses the following terminology. An occurrence of a term of the form $f(\mathbf{t})$ in a rule is *critical* if

- it is not the right-hand side of any equation,
- the function symbol f is indefinite, and
- there are no indefinite function symbols in $\mathbf{t}$.

In rule (2), for example, X/Y is critical. In the fact $p(1\,..\,8)$, $1\,..\,8$ is critical. In the rule

$$q(1\,..\,(X/2)) \leftarrow p(X) \quad (12)$$

$X/2$ is critical, and $1\,..\,(X/2)$ is not. Furthermore, let t be either a basic symbol different from numerals or a term of the form $c(\mathbf{t})$, where c is a symbolic constant. About an occurrence of t in a rule we say that it is *problematic* if it is in the scope of an arithmetic function symbol, and that it is *nasty* if it is the left-hand side of an equation that includes an indefinite arithmetic function symbol. For example, n in $p(n+1)$ and $n(3)$ in $p(n(3)+1)$ are problematic; n in the rule

$$p(X) \leftarrow n = X/2 \quad (13)$$

is nasty.

A rule in normal form does not contain term occurrences of any of these three types. Indeed, condition (a) in the definition of normal form eliminates problematic occurrences, and condition (b) eliminates both critical and nasty occurrences.

The normal form of a rule is constructed by the following algorithm:

while the rule contains a problematic occurrence, apply to it transformation (11);

while the rule contains a critical occurrence, apply to it transformation (11);

while the rule contains a nasty occurrence, apply to it transformation (11).

In case of the fact $p(n+1)$, executing the first loop gives its normal form (5). For rule (12), executing the body of the second loop gives

$$q(1\,..\,Y) \leftarrow p(X) \wedge Y = X/2,$$

and after the second execution we get the normal form

$$q(Z) \leftarrow p(X) \wedge Y = X/2 \wedge Z = 1\,..\,Y.$$

Rule (13) is converted to its normal form

$$p(X) \leftarrow Y = X/2 \wedge Y = n$$

by the third loop.

The algorithm above is guaranteed to terminate. Indeed, executing the body of the first loop moves a problematic occurrence to a position that is not in the scope of an arithmetic function symbol; hence the number of problematic occurrences goes down. After executing the body of the second loop, a critical occurrence becomes the right-hand side of an equation; hence the number of critical occurrences goes down. After executing the body of the third loop, a nasty occurrence becomes the right-hand side of an equation; hence the number of nasty occurrences goes down.

To prove that the rule produced by this algorithm is in normal form, consider the intermediate results generated in the process of applying the algorithm to a rule R. Let R_1 be the result of executing the first loop, let R_2 be the result of executing the second loop, and let R_3 be the output. There are no problematic occurrences of terms in R_1. It follows that there are no such occurrences in R_2 and R_3, because problematic occurrences cannot be introduced by transformation (11). Hence R_3 satisfies condition (a).

To prove that the rule R_3 satisfies condition (b), we will show first that every occurrence of a term of the form $f(\mathbf{t})$ with indefinite f in that rule is the right-hand side of an equation. Assume that some occurrence of $f(\mathbf{t})$ with indefinite f is not the right-hand side of any equation. Take a subterm of $f(\mathbf{t})$ that is minimal among subterms containing an indefinite function symbol. It has the form $f'(\mathbf{t}')$ with indefinite f' and no other indefinite function symbols. This subterm is not the right-hand side of any equation, so that it is critical. But there are no critical occurrences in R_3, because there are no critical occurrences in R_2, and critical occurrences cannot be introduced by transformation (11).

It remains to check that the equation with the right-hand side $f(\mathbf{t})$ is numeric. Every basic symbol in $f(\mathbf{t})$ is a numeral, because R_3 satisfies condition (a). Assume that the left-hand side t of the equation contains a basic symbol that is not a numeral. Since R_3 satisfies condition (a), the outermost symbol of t is not an arithmetic function symbol. Hence t is either a basic symbol different from numerals or a term of the form $c(\mathbf{t})$, where c is a symbolic constant. This is impossible, because R_3 does not contain nasty occurrences.

The theorem below describes the relationship between a rule and its normal form. By *Int*(D) we denote the many-sorted intuitionistic predicate calculus with equality (Fandinno and Lifschitz 2023, Section 5.1) over the signature $\sigma(D)$.

Theorem 1. *If N is the normal form of a rule R then $\tau^*(N)$ is equivalent to $\tau^*(R)$ in* Int(D).

5 Generalized natural translation

Consider a rule N in normal form. Applying the translation ν to N involves substituting an integer variable for each variable that occurs in N at least once

- in the scope of an arithmetic function symbol, or
- in the left-hand side of an equation that contains an indefinite function symbol in the right-hand side.

Make the list $V_1, \dots, V_m$ of all variables satisfying this condition, and choose m distinct integer variables $I_1, \dots, I_m$. For any mini-GRINGO term t that occurs in N and does not contain indefinite function symbols, the result of

(i) substituting $I_1, \dots, I_m$ for $V_1, \dots, V_m$ and

(ii) replacing every subexpression of the form $c(r_1, \dots, r_k)$, where c is a symbolic constant, with $(c\backslash k)(r_1, \dots, r_k)$

is a term over the signature $\sigma(D)$. The operator that performs this transformation will be denoted by $p2f$. It applies to tuples of terms componentwise.

For example, if N is rule (1) then $m = 1$, V_1 is Y, and $p2f(X, Y + 1)$ is $(X, I_1 + 1)$. If N is rule (4) then $m = 3$, $p2f(X, Y)$ is (I_1, I_2), and $p2f(Z + 1)$ is $I_3 + 1$.

The translation ν is defined first for expressions that may occur in the head and the body of N:

- If $\mathbf{t}$ is a k-tuple of terms that do not contain indefinite function symbols then
  - $\nu(p(\mathbf{t}))$ is $(p/k)(p2f(\mathbf{t}))$,
  - $\nu(\mathit{not}\ p(\mathbf{t}))$ is $\neg(p/k)(p2f(\mathbf{t}))$,
  - $\nu(\mathit{not\ not}\ p(\mathbf{t}))$ is $\neg\neg(p/k)(p2f(\mathbf{t}))$.
- If $t_1 \prec t_2$ is a comparison that does not contain indefinite function symbols then $\nu(t_1 \prec t_2)$ is $p2f(t_1) \prec p2f(t_2)$,
- $\nu(t = f(t_1, \dots, t_k))$, where f is an indefinite function symbol, is

$$F\big(p2f(t_1), \dots, p2f(t_k), p2f(t)\big),$$

  where $F(N_1, \dots, N_{k+1})$ is a formula defining f.
- The result of applying ν to the empty string is $\bot$ (false).

The result of applying the translation ν to a rule (7) in normal form is defined as the sentence

$$\widetilde{\forall}(\nu(B_1) \wedge \cdots \wedge \nu(B_n) \rightarrow \nu(H)). \tag{14}$$

For example, ν transforms rule (1) into

$$\forall X I_1(p(X, I_1) \rightarrow q(X, I_1 + 1)), \tag{15}$$

rule (5) into

$$\forall I_1(I_1 = n \rightarrow p(I_1)),$$

and rule (6) into

$$\forall I_1(1 \leq I_1 \leq 8 \to p(I_1)).$$

The theorem below shows that the translations ν and τ^* are equivalent whenever the former is applicable. It is similar to the main result of the original natural translation paper (Lifschitz 2021) and is proved in a similar way.

Theorem 2. *For any rule N in normal form, the formula $\nu(N)$ is equivalent to $\tau^*(N)$ in* Int(D).

6 Making the Translation Completable

A sentence over $\sigma(D)$ is *completable* if it has the form

$$\widetilde{\forall}(F \to p(\mathbf{V})), \tag{16}$$

where $\mathbf{V}$ is a list of pairwise distinct variables of the sort general. This is a special case of the definition used by Fandinno, Lifschitz, and Temple (2024) for extending Clark's definition of program completion (Clark 1978) to mini-gringo programs. Applying the transformation τ^* to any non-constraint rule produces a completable sentence. Though the output of ν is not always a completable sentence, we can easily apply intuitionistically equivalent transformations to make it so.

The result of applying ν to a non-constraint rule is always of the form

$$\widetilde{\forall}(F \to p(\mathbf{t})), \tag{17}$$

where $\mathbf{t}$ is a list $t_1, \dots, t_k$ of terms. Any sentence of this form can be converted to a completable sentence as follows. Pick a list $V_1, \dots, V_k$ of pairwise distinct variables of the sort general that do not occur in F or $t_1, \dots, t_k$. Then, for each $i = 1, \dots, k$, if t_i is not a general variable different from $t_1, \dots, t_{i-1}$, then

- replace t_i in the consequent by V_i;
- add the conjunctive term $V_i = t_i$ to the antecedent;
- add $\forall V_i$ to the list of quantifiers in front of the implication.

Clearly, after applying this transformation to all terms t_i, we obtain a sentence where all the terms in the consequent are pairwise distinct variables of the sort general; hence, the result is a completable sentence. It is also clear that each step is an intuitionistically equivalent transformation, so that the resulting sentence is equivalent to sentence (17) in *Int*(D).

For example, rule (1) is transformed by ν into sentence (15), which is not completable because the term $I_1 + 1$ occurring in its head is not a variable. Applying the transformation above to (15), we obtain the completable sentence

$$\forall X V_2 I_1(p(X, I_1) \wedge V_2 = I_1 + 1 \to q(X, V_2)), \tag{18}$$

To give an example of using the completable sentences produced by the algorithm above for forming the completed definition of a predicate (Fandinno, Lifschitz, and Temple 2024, Section 3.1), consider the pair of rules: (1) and

$$q(X, Y) \leftarrow r(Y, X). \tag{19}$$

Rule (19) is in normal form, and the result of applying the transformation ν to it is the completable formula

$$\forall XY(r(Y, X) \to q(X, Y)). \tag{20}$$

If a program defines the predicate $q/2$ by rules (1) and (19), then the completed definition of this predicate can be constructed in two steps. First, we rename bound variables so that the consequents of two implications become identical. For example, we rename variable V_2 in (18) to Y, obtaining:

$$\forall XY I_1(p(X, I_1) \wedge Y = I_1 + 1 \to q(X, Y)).$$

Then we form an equivalence expressing that, jointly, these two sufficient conditions for $q/2$ give a necessary condition:

$$\forall XY(q(X, Y) \leftrightarrow \exists I_1(p(X, I_1) \wedge Y = I_1 + 1) \vee r(Y, X)).$$

7 Proof of Theorem 1

In the following, we say that two formulas F and G are *equivalent* if the universal closure of $F \leftrightarrow G$ is provable in $Int(D)$. We say that two rules are equivalent if their translations by τ^* are equivalent formulas.

To prove Theorem 1, it is sufficient to show that applying procedure (11) to the any rule results in an equivalent rule (Lemma 4 below).

We start by introducing some notation. We use $E[V]$ to indicate that E is a formula, term, or other syntactic entity contains at most one occurrence of a variable V. Then, by $E[t]$ we denote the result of replacing that occurrence of V in $E[V]$ by t.

Lemma 1. *Let $r[V]$ be a term with exactly one occurrence of a variable V, and t be a term that does not contain V. Then,*

$$val_{r[t]}(W) \leftrightarrow \exists V\,(val_{r[V]}(W) \wedge val_t(V)) \tag{21}$$

is provable in $Int(D)$.

Proof. By induction on the structure of $r[V]$. *Base case:* Since V occurs in $r[V]$, the latter cannot be a symbolic constant. Hence $r[V]$ is the variable V itself, and (21) is

$$val_t(W) \leftrightarrow \exists V\,(W = V \wedge val_t(V)), \tag{22}$$

which, since V does not occur in t, is provable in $Int(D)$.

Induction case 1: $r[V]$ is of the form $c(r_1, \dots, r_k)$ where c is a symbolic constant and V is a subterm of $r_i[V]$ for some $i \in \{1, \dots, n\}$. Assume without loss of generality that $i = 1$. Then, the left-hand and right-hand side of (21) respectively are

$$\exists \mathbf{Y}\,(val_{r_1[t]}(Y_1)\ \wedge G \wedge W = P) \tag{23}$$

$$\exists V \exists \mathbf{Y}\,(val_{r_1[V]}(Y_1) \wedge G \wedge W = P \wedge val_t(V)) \tag{24}$$

where $\mathbf{Y} = Y_1, \dots, Y_k$ is a list of fresh variables, G stands for

$$val_{r_2}(Y_2) \wedge \dots \wedge val_{r_k}(Y_k)$$

and P stands for $(c\backslash k)(Y_1, \dots, Y_k)$. Since V occurs neither in G nor in P, formula (24) is equivalent to

$$\exists \mathbf{Y}\,(\exists V\,(val_{r_1[V]}(Y_1) \wedge val_t(V)) \wedge G \wedge W = P) \tag{25}$$

By induction hypothesis,

$$val_{r_1[t]}(W_1) \leftrightarrow \exists V\,(val_{r_1[V]}(W_1) \wedge val_t(V))$$

is provable in $Int(D)$ and, thus, (23) and (25) are equivalent. Hence, the left-hand and right-hand side of (21) are equivalent. *Induction case 2:* $r[V]$ is of the form $f(r_1, \ldots, r_k)$ where f is a definite arithmetic function symbol, then the left-hand and right-hand side of (21) respectively are

$$\exists \mathbf{Y}\,(val_{r_1[t]}(Y_1)\ \wedge G \wedge W = f(Y_1, \ldots, Y_k))$$
$$\exists V \exists \mathbf{Y}\,(val_{r_1[V]}(Y_1) \wedge G \wedge W = f(Y_1, \ldots, Y_k) \wedge val_t(V))$$

where G is as above and we similarly assume that V is a subterm of $r_1[V]$. The rest of the proof is analogous to Induction case 1. *Induction case 3:* $r[V]$ is of the form $f(r_1, \ldots, r_k)$ where f is an indefinite arithmetic function symbol, then the left-hand and right-hand side of (21) respectively are

$$\exists \mathbf{Y}\,(val_{r_1[t]}(Y_1)\ \wedge G \wedge I_{k+1} = W \wedge F(I_1, \ldots, I_{k+1}))$$
$$\exists V \exists \mathbf{Y}\,(val_{r_1[V]}(Y_1) \wedge G \wedge I_{k+1} = W \wedge F(I_1, \ldots, I_{k+1}) \wedge val_t(V))$$

The rest of the proof is analogous to Induction case 1. □

Lemma 2. *If t is a term and V is a variable, then*

$$\tau^B(V = t) \leftrightarrow val_t(V)$$

is provable in $Int(D)$.

Proof. Note that $\tau^B(V = t)$ is

$$\exists X_1 \exists X_2 (X_1 = V \wedge val_t(X_2) \wedge X_1 = X_2)$$

where X_1 and X_2 are fresh variables. This is equivalent to

$$\exists X_1 \exists X_2 (X_1 = V \wedge val_t(V) \wedge V = X_2)$$

which is equivalent to $val_t(V)$ because neither X_1 nor X_2 occur in $val_t(V)$. □

Lemma 3. *If $L[V]$ is a mini-gringo literal containing exactly one occurrence of V and t is a term that does not contain V, then*

$$\tau^B(L[t]) \leftrightarrow \exists V(\tau^B(L[V]) \wedge \tau^B(V = t)) \qquad (26)$$

is provable in $Int(D)$.

Proof. By case analysis on the form of $L[V]$. *Case 1:* L is of the form $c(r_1, \ldots, r_k)$ where t is a subterm of r_i for some i. Assume without loss of generality that $i = 1$. Hence, the left-hand and right-hand side of (26) respectively are

$$\exists \mathbf{Y}\,(val_{r_1[t]}(Y_1)\ \wedge G \wedge P) \qquad (27)$$
$$\exists V(\exists \mathbf{Y}\,(val_{r_1[V]}(Y_1) \wedge G \wedge P) \wedge \tau^B(V = t) \qquad (28)$$

where G and P are as in the proof of Lemma 1. By Lemma 2, formula (28) is equivalent to

$$\exists V(\exists \mathbf{Y}\,(val_{r_1[V]}(Y_1) \wedge G \wedge P) \wedge val_t(V)) \qquad (29)$$

Furthermore, since none of $Y_1, \ldots, Y_k$ occur free in $val_t(V)$ and V occur neither in G nor P, formula (29) is equivalent to

$$\exists \mathbf{Y}\,(\exists V(val_{r_1[V]}(Y_1) \wedge val_t(V)) \wedge G \wedge P) \qquad (30)$$

By Lemma 1, formulas (27) and (30) are equivalent. The other cases are similar. □

Lemma 4. *For any mini-gringo rule of the form*

$$Head[V] \leftarrow Body[V]$$

with exactly one occurrence of variable V and any term t that does not contain V, the equivalence between sentences

$$\tau^*(Head[t]\ \leftarrow Body[t]) \qquad (31)$$
$$\tau^*(Head[V] \leftarrow Body[V] \wedge V = t) \qquad (32)$$

is provable in $Int(D)$.

Proof. We proceed by cases based on the position of the single occurrence of V. Assume first that the single occurrence of V is in $Head[V]$. Then, $Body[V]$ and $Body[t]$ are identical, and $Head[V]$ is of the form $c(r_1, \ldots, r_k)$ with V occurring in $r_i[V]$ for some $i \in \{1, \ldots, k\}$. Assume without loss of generality that $i = 1$. Sentences (31) and (32) can be respectively written as the universal closures of formulas

$$val_{r_1[t]}(Y_1) \wedge B \rightarrow H \qquad (33)$$
$$val_{r_1[V]}(Y_1) \wedge B \wedge \tau^B(V = t) \rightarrow H \qquad (34)$$

where H stands for $(c/k)(Y_1, \ldots, Y_k)$ and B stands for $\tau^B(Body[V]) \wedge G$ with G as in the proof of Lemma 1. Since the only occurrence of V in the rule is in $r_1[V]$, it follows that V does not occur in B nor H. Hence, (34) is equivalent to

$$\exists V(val_{r_1[V]}(Y_1) \wedge \tau^B(V = t)) \wedge B \rightarrow H \qquad (35)$$

which, by Lemma 2, is equivalent to

$$\exists V(val_{r_1[V]}(Y_1) \wedge val_t(V)) \wedge B \rightarrow H \qquad (36)$$

By Lemma 1, this is equivalent to (33).

Assume now that the single occurrence of V is in $Body[V]$. Then, $Head[V]$ and $Head[t]$ are identical and $Body[V]$ is of the form $L_1 \wedge \cdots \wedge L_n$ with V occurring in $L_i[V]$ for some $i \in \{1, \ldots, n\}$. Assume without loss of generality that $i = 1$. Sentences (31) and (32) can be respectively written as the universal closures of formulas

$$\tau^B(L_1[t]) \wedge B \rightarrow H \qquad (37)$$
$$\tau^B(L_1[V]) \wedge B \wedge \tau^B(V = t) \rightarrow H \qquad (38)$$

where H is as above, and B stands for

$$\tau^B(L_2) \wedge \cdots \wedge \tau^B(L_n).$$

Since the only occurrence of V in the rule is in $L_i[V]$, it follows that V does not occur in B nor H. Hence, (38) is equivalent to

$$\exists V(\tau^B(L_1) \wedge \tau^B(V = t)) \wedge B \rightarrow H$$

which, by Lemma 3, is equivalent to (37). □

8 Proof of Theorem 2

If $\mathbf{t}$ and $\mathbf{t}'$ respectively are lists $t_1, \ldots, t_m$ and $t'_1, \ldots, t'_m$ of terms of the same length, then $\mathbf{t} = \mathbf{t}'$ stands for the conjunction of equalities

$$t_1 = t'_1 \wedge \cdots \wedge t_m = t'_m$$

We assume that N is a rule (7) in normal form, and that C is

$$I_1 = V_1 \wedge \cdots \wedge I_m = V_m,$$

where $V_1, \ldots, V_m, I_1, \ldots, I_m$ are variables as in the definition of *p2f* (Section 5).

Lemma 5. *For any term t that occurs in N in the scope of an arithmetic function or in an equation of the form $t = t'$ such that t' contains an indefinite function symbol, $p2f(t)$ is of the sort integer.*

Proof. By cases. *Case 1:* t is a variable. Since N is in normal form and t occurs in the scope of a definite arithmetic function symbol or in an equation of the form $t = t'$ such that t' contains an indefinite function symbol, it follows that t is one of the variables V_k ($1 \leq k \leq m$). Then by definition, $p2f(V_k)$ is the integer variable I_k.

Case 2: t is a basic symbol. By definition of normal form, t must be a numeral. Then, $p2f(t)$ is t, and its sort is integer.

Case 3: t is of the form $f(t_1, \dots, t_k)$. From the condition (b) of the definition of normal form, it follows that f is a definite arithmetic function symbol. Then, by definition, $p2f(f(t_1, \dots, t_k))$ is $f(p2f(t_1), \dots, p2f(t_k))$, which is of the sort integer. □

Lemma 6. *For any tuple $\mathbf{t}$ of terms that occur in N and contain no occurrences of indefinite function symbols, and any tuple $\mathbf{V}$ of variables of the same length as $\mathbf{t}$, the formulas*

(i) $C \to \forall \mathbf{V}(val_{\mathbf{t}}(\mathbf{V}) \leftrightarrow \mathbf{V} = p2f(\mathbf{t}))$;
(ii) $C \to (\nu(p(\mathbf{t})) \leftrightarrow \forall \mathbf{V}(val_{\mathbf{t}}(\mathbf{V}) \to (p/k)(\mathbf{V})))$, *where k is arity of $\mathbf{t}$;*
(iii) $C \to (\nu(p(\mathbf{t})) \leftrightarrow \tau^B(p(\mathbf{t})))$;
(iv) $C \to (\nu(not\ p(\mathbf{t})) \leftrightarrow \tau^B(not\ p(\mathbf{t})))$;
(v) $C \to (\nu(not\ not\ p(\mathbf{t})) \leftrightarrow \tau^B(not\ not\ p(\mathbf{t})))$

are provable $Int(D)$.

Proof. (i) It is sufficient to consider the case when $\mathbf{t}$ is a single term t, so that the formula to be proven is

$$C \to \forall V(val_t(V) \leftrightarrow V = p2f(t)). \tag{39}$$

The proof is by induction on t. *Case 1:* t is one of the variables V_k ($1 \leq k \leq m$). Then the consequent of (39) is $\forall V(V = V_k \leftrightarrow V = I_k)$, and the antecedent C contains the conjunctive term $V_k = I_k$.

Case 2: t is a variable different from $V_1, \dots, V_m$, or a basic symbol. Then, the consequent of (39) is

$$\forall V(V = t \leftrightarrow V = t).$$

Case 3: t is $c(\mathbf{t})$, where c is a symbolic constant and $\mathbf{t}$ is a list $t_1, \dots, t_k$ of terms. Then, $val_t(V)$ is

$$\exists \mathbf{Y}(val_{\mathbf{t}}(\mathbf{Y}) \wedge V = (c\backslash k)(\mathbf{Y}))$$

where $\mathbf{Y}$ is a list $Y_1, \dots, Y_k$ of general variables not occurring in t of the same length as $\mathbf{t}$.

By the induction hypothesis, under the assumption C, the formula above can be simplified as follows:

$$\exists \mathbf{Y}(\mathbf{Y} = p2f(\mathbf{t}) \wedge V = (c\backslash k)(\mathbf{Y}))$$

which is equivalent to

$$V = (c\backslash k)(p2f(\mathbf{t}))$$

that is $V = p2f(c(\mathbf{t}))$.

Case 4: t is $f(\mathbf{t})$, where f is a definite arithmetic function symbol and $\mathbf{t}$ is a list $t_1, \dots, t_k$ of terms. Then, $val_t(V)$ is

$$\exists \mathbf{J}(val_{\mathbf{t}}(\mathbf{J}) \wedge V = f(\mathbf{J}))$$

where $\mathbf{J}$ is a list $J_1, \dots, J_k$ of integer variables. Then, by the induction hypothesis, under the assumption C, the formula above can be simplified as follows:

$$\exists \mathbf{J}(\mathbf{J} = p2f(\mathbf{t}) \wedge V = f(\mathbf{J}))$$

By Lemma 5 each of the terms in $p2f(\mathbf{t})$ is of the sort integer. Hence, the above formula is equivalent to

$$V = f(p2f(\mathbf{t}))$$

We conclude that

$$val_{f(\mathbf{t})}(V) \leftrightarrow V = f(p2f(\mathbf{t}))$$

By the definition, $f(p2f(\mathbf{t}))$ is $p2f\big(f(\mathbf{t})\big)$.

Items (ii)-(v) follow as in the proof of Lemma 1 (ii)-(v) by Lifschitz (2021). □

Lemma 7. *For any comparison $t_1 \prec t_2$ that occurs in N and contains no occurrences of indefinite function symbols, the formula*

$$C \to (\nu(t_1 \prec t_2) \leftrightarrow \tau^B(t_1 \prec t_2))$$

is provable in $Int(D)$.

Proof. The proof follows using Lemma 6(i) as the proof of Lemma 2 by Lifschitz (2021). □

In the following lemmas, by $\mathbf{J}$ we denote a list $J_1, \dots, J_k$ of integer variables.

Lemma 8. *Let f be an indefinite function symbol, and let $F(\mathbf{J}, K)$ be a formula over $\sigma^-(D)$ defining f. For any term $f(\mathbf{t})$ occurring in N and every term r of the sort integer, formula*

$$C \to (val_{f(\mathbf{t})}(r) \leftrightarrow F(p2f(\mathbf{t}), r)) \tag{40}$$

is provable in $Int(D)$.

Note that substituting $p2f(\mathbf{t})$ for $\mathbf{J}$ in $F(\mathbf{J}, K)$ is allowed because, by Lemma 5, $p2f(\mathbf{t})$ is a tuple of integer terms.

Proof. The left-hand side of the equivalence in (40) stands for

$$\exists \mathbf{J} K(val_{\mathbf{t}}(\mathbf{J}) \wedge K = r \wedge F(\mathbf{J}, K))$$

which, since r is of the sort integer, is equivalent to

$$\exists \mathbf{J}(val_{\mathbf{t}}(\mathbf{J}) \wedge F(\mathbf{J}, r)).$$

By condition (b) of the normal form definition it follows that $\mathbf{t}$ are terms containing no occurrences of indefinite function symbols. By Lemma 6(i), this formula can be rewritten, under the assumption C, as

$$\exists \mathbf{J}(\mathbf{J} = p2f(\mathbf{t}) \wedge F(\mathbf{J}, r))$$

which is equivalent to the right-hand side of the equivalence in (40). □

Lemma 9. *For any numeric equation $t = f(\mathbf{t})$ occurring in N where f is an indefinite function symbol, the formula*

$$C \to (\nu(t = f(\mathbf{t})) \leftrightarrow \tau^B(t = f(\mathbf{t}))$$

is provable in $Int(D)$.

Proof. By definition, $\tau^B(t = f(\mathbf{t}))$ is

$$\exists X_1 X_2 \big(val_t(X_1) \wedge val_{f(\mathbf{t})}(X_2) \wedge X_1 = X_2\big)$$

where X_1 and X_2 are general variables. This formula is equivalent to formula

$$\exists X \big(val_t(X) \wedge val_{f(\mathbf{t})}(X)\big).$$

By Lemma 6(i), the last formula can be rewritten, under the assumption C, as

$$\exists X \big(X = p2f(t) \wedge val_{f(\mathbf{t})}(X)\big),$$

which can be further rewritten as

$$val_{f(\mathbf{t})}(p2f(t)).$$

By Lemma 5, $p2f(t)$ is an integer term. Then, by Lemma 8, the last formula is equivalent, under the assumption C, to

$$F(p2f(\mathbf{t}), p2f(t)).$$

This is $\nu(t = f(\mathbf{t}))$. □

Lemma 10. *If a term t occurs in a rule in N and V is a variable that occurs in t at least once in the scope of an arithmetic function symbol, then formula*

$$\exists W\, val_t(W) \to \exists I(I = V),$$

where W is a general variable, is provable in $Int(D)$.

Proof. Let us assume first that t is an expression of the form $f(\mathbf{t})$ such that some t_i is V where $\mathbf{t}$ is a list $t_1, \dots, t_k$ of terms. We proceed by cases on whether f is definite or indefinite. *Case 1.* f is a definite arithmetic function. Then the antecedent of the implication to be proved is

$$\exists W \mathbf{J}(val_{\mathbf{t}}(\mathbf{J}) \wedge W = f(\mathbf{J})).$$

and

$$\exists J_i\, val_{t_i}(J_i) \quad = \quad \exists J_i\, val_V(J_i) \quad = \quad \exists J_i (V = J_i)$$

The last formula is equivalent to $\exists I(I = V)$, which is the consequent of the formula to be proved and is implied by $val_{\mathbf{t}}(\mathbf{J})$. *Case 2.* The reasoning is similar to Case 1. The general case follows by induction on the size of t because there is a minimal subterm of the form $f(\mathbf{t})$ such that some t_i is V. □

Lemma 11. *If B is a literal or a comparison occurring in N that does not contain indefinite function symbols, and V is a variable that occurs in B at least once in the scope of an arithmetic function symbol, then formula*

$$\tau^B(Body) \to \exists I(I = V)$$

is provable in $Int(D)$.

The proof follows the proof of Lemma 7 by Lifschitz (2021), but using Lemma 10 in this paper in place of Lemma 6 used there.

Lemma 12. *For any term t whose only basic symbols are numerals, and any variable V occurring in t, formula*

$$\exists N\, val_t(N) \to \exists I(I = V)$$

is provable in $Int(D)$.

Proof. By induction on t. Since all basic symbols occurring in t are numerals, three cases are possible. *Case 1:* Term t is V. Then the antecedent $\exists N\, val_t(N)$ of the implication to be proven is $\exists N(N = V)$, which is equivalent to its consequent. *Case 2:* Term t has the form $f(t_1, \dots, t_k)$, where f is a definite arithmetic function symbol. Then the antecedent $\exists N\, val_t(N)$ of the implication to be proven is

$$\exists N \mathbf{J}(val_{\mathbf{t}}(\mathbf{J}) \wedge N = f(\mathbf{J})).$$

where $\mathbf{J}$ is a list of integer variables. Then, V occurs in some term t_i. The formula above implies $\exists I\, val_{t_i}(I)$. By the induction hypothesis, $\exists I(I = V)$ follows. *Case 3:* Term t has the form $f(t_1, \dots, t_k)$, where f is a indefinite arithmetic function symbol. Similar to Case 2. □

Lemma 13. *If B is a numeric equation of the form $t = f(\mathbf{t})$ such that f is an indefinite arithmetic function and V is a variable that occurs in B at least once, then formula*

$$\tau^B(Body) \to \exists I(I = V)$$

is provable in $Int(D)$.

Proof. The antecedent $\tau^B(Body)$ of the formula to be proved is

$$\exists X_1 X_2 (val_t(X_1) \wedge val_{f(\mathbf{t})}(X_2) \wedge X_1 = X_2),$$

which is equivalent to

$$\exists X (val_t(X) \wedge val_{f(\mathbf{t})}(X)) \tag{41}$$

Case 1: V occurs in t. Formula (41) can be expanded as

$$\exists X (val_t(X) \wedge \exists \mathbf{J} K (val_{\mathbf{t}}(\mathbf{J}) \wedge K = X \wedge F(\mathbf{J}, K))).$$

The above sentence implies $\exists K\, val_t(K)$, and hence $\exists I(I = V)$ follows by Lemma 12, because all basic symbols occurring in t are numerals. *Case 2:* V occurs in $f(\mathbf{t})$. Formula (41) implies $\exists X\, val_{f(\mathbf{t})}(X)$, and hence $\exists I(I = V)$ follows by Lemma 10. □

Lemma 14. *Let $\mathbf{t}$ be a list of terms occurring in N, let $V_1, \dots, V_k$ be all the variables that occur in $\mathbf{t}$ in the scope of an arithmetic function symbol, $\mathbf{Y}$ be a list of general variables of the same length k and c be a symbolic constant. Then, sentences*

$$\forall \mathbf{Y}(val_{\mathbf{t}}(\mathbf{Y}) \to (c/k)(\mathbf{Y})) \tag{42}$$

$$\bigwedge_{i=1}^{k} \exists I(I = V_i) \to \forall \mathbf{Y}(val_{\mathbf{t}}(\mathbf{Y}) \to (c/k)(\mathbf{Y})) \tag{43}$$

are equivalent in $Int(D)$, where I is a integer variable.

Proof. It is sufficient to show that, for every i,

$$\exists I(I = V_i) \to \forall \mathbf{Y}(val_{\mathbf{t}}(\mathbf{Y}) \to (c/k)(\mathbf{Y})) \tag{44}$$

is equivalent to (42). This is equivalent to

$$\forall \mathbf{Y}(\exists I(I = V_i) \wedge val_{\mathbf{t}}(\mathbf{Y}) \to (c/k)(\mathbf{Y})). \tag{45}$$

By Lemma 10, the conjunction in the antecedent is equivalent to to its second conjunctive term. □

Proof of the Theorem 2. We need to show that formulas (14) and

$$\tau^*(H \leftarrow B_1 \wedge \cdots \wedge B_n) \tag{46}$$

are equivalent to each other in $Int(D)$. Let $Body$ stand for the conjunction $B_1 \wedge \cdots \wedge B_n$ so that $\tau^B(Body)$ and $\nu(Body)$ respectively are

$$\tau^B(B_1) \wedge \cdots \wedge \tau^B(B_n) \quad \text{and} \quad \nu(B_1) \wedge \cdots \wedge \nu(B_n)$$

Let H be an atom $c(\mathbf{t})$, where $\mathbf{t}$ is a tuple $t_1, \dots, t_k$ (the case when H is empty is similar). Then, formula (46) has the form

$$\widetilde{\forall}(val_{\mathbf{t}}(\mathbf{Y}) \wedge \tau^B(Body) \to (c/k)(\mathbf{Y})),$$

which is equivalent to

$$\widetilde{\forall}\left(\tau^B(Body) \to \forall \mathbf{Y}(val_{\mathbf{t}}(\mathbf{Y}) \to (c/k)(\mathbf{Y}))\right). \tag{47}$$

By Lemma 14, it follows that the consequent of (47) is equivalent to

$$\bigwedge_{i=1}^{k} \exists I(I = V_i) \to \forall \mathbf{Y}(val_{\mathbf{t}}(\mathbf{Y}) \to (c/k)(\mathbf{Y})) \tag{43}$$

where $V_1, \dots, V_k$ are all the variables occurring in $\mathbf{t}$ in the scope of some arithmetic function symbol. Then the variables $V_{k+1}, \dots, V_m$ occur in the scope of an arithmetic function symbol in $Body$. By Lemmas 11 and 13 it follows that formula

$$\tau^B(Body) \to \exists I(I = V_i)$$

is provable in $Int(D)$ for every i with $k+1 \leq i \leq m$. Hence the antecedent $\tau^B(Body)$ of (47) is equivalent to

$$\bigwedge_{i=k+1}^{m} \exists I(I = V_i) \wedge \tau^B(Body)$$

and, thus, (47) is equivalent to

$$\widetilde{\forall}\left(\bigwedge_{i=1}^{m} \exists I(I = V_i) \wedge \tau^B(Body) \to F\right),$$

This can be rewritten as

$$\widetilde{\forall}(C \to (\tau^B(Body) \to F)). \tag{48}$$

From Lemmas 6(i,iii,iv,v), 7, 9 we can conclude that formula (48) is equivalent to

$$\widetilde{\forall}(C \to (\nu(Body) \to \forall \mathbf{Y}(\mathbf{Y} = p2f(\mathbf{t}) \to (c/k)(\mathbf{Y})))),$$

which is equivalent to

$$\widetilde{\forall}(C \to (\nu(Body) \to \nu(H))).$$

The only part of the last formula that contains any of the variables V_i is C. Consequently that formula is equivalent to

$$\widetilde{\forall}\left(\bigwedge_i \exists V_i(I_i = V_i) \to (\nu(Body) \to \nu(H))\right).$$

Since the antecedent $\bigwedge_i \exists V_i(I_i = V_i)$ is provable in intuitionistic logic, it can be dropped, which leads us to formula (14). □

9 Conclusion

This paper describes an approach to transforming mini-gringo rules into first-order sentences that involves a preprocessing step – converting the rule to a normal form. We expect that the new translation will be implemented in a future version of the proof assistant ANTHEM, and that it will make ANTHEMeasier to use. We plan to extend this translation method to answer set programming languages that include useful programming constructs not available in mini-gringo, such as conditional literals and aggregates.

Acknowledgements

We would like to thank Michael Gelfond, Zachary Hansen and Tobias Stolzmann for their comments on a draft of this paper.

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