# Stable Models for Infinitary Formulas with Extensional Atoms

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**Abstract**

The definition of stable models for propositional formulas with infinite conjunctions and disjunctions can be used to describe the semantics of answer set programming languages. In this note, we enhance that definition by introducing a distinction between intensional and extensional atoms. The symmetric splitting theorem for first-order formulas is then extended to infinitary formulas and used to reason about infinitary definitions.

## 1 Introduction

The original definition of a stable model (Gelfond and Lifschitz, 1988) was applicable only to quantifier-free formulas of a restricted syntax. Stable models for arbitrary first-order sentences were defined by Ferraris et al. (2007) using the stable model operator SM. This definition can be used to define the semantics of some rules with aggregate expressions. For instance, the following rule, written in the input language of the ASP system CLINGO,[1]

```
q :- #count{X:p(X)} = 0                (1)
```

can be identified with the first-order sentence

$$\forall x\, \neg p(x) \to q. \tag{2}$$

In Ferraris et al. (2011), that definition was generalized in a different direction: allowing a distinction between "extensional" and "intensional" predicate symbols. (Under the original definition all predicate symbols are treated as intensional.) Intuitively, an intensional predicate is one whose extent is defined by the program, while all other, extensional, predicates are defined externally. Similar distinctions have been proposed many times: Gelfond and Przymusinska (1996) distinguish between input and output predicates in their "lp-functions", Oikarinen and Janhunen (2008) distinguish between input and output atoms, and Lierler and Truszczynski (2011) between input and non-input atoms. These distinctions are useful because they allow for a modular view of logic programs. For example, in the splitting theorem from Ferraris et al. (2009), the authors showed that stable models for a program can sometimes be computed by breaking the program

[1] HTTP://POTASSCO.SOURCEFORGE.NET

into parts and computing the stable models of each part separately using different sets of intensional predicates.

Using the approach proposed by Ferraris (2005), Truszczynski (2012) extended the definition of a stable model in a different direction: he showed how to apply this concept to infinitary propositional formulas. He also showed that the definition of first-order stable models in terms of the 2007 definition of the operator SM could be reduced to the definition of infinitary stable models. Infinitary stable models were used in that paper as a tool for relating first-order stable models to the semantics of first-order logic with inductive definitions. Infinitary stable models are important also because they provide an alternative understanding of the semantics of aggregates. For instance, rule (1) can be identified with the infinitary formula

$$\bigwedge_t \neg p(t) \rightarrow q, \tag{3}$$

where the conjunction in the antecedent is understood as ranging over all ground terms $t$ not containing arithmetic operations. The advantage of this approach over the use of first-order formulas is that it is more flexible. For example, it is applicable to aggregates involving `#sum`. In recent work, Gebser et al. (2015) use this idea to define a precise semantics for a large class of ASP programs, including programs with local variables and aggregate expressions.

However, Truszczynski's definition of stable models for infinitary formulas does not allow a distinction between extensional and intensional atoms. It treats all atoms as intensional. In this note, we generalize the definition of stable models for infinitary formulas to accommodate both intensional and extensional atoms, and we study properties of this definition. As might be expected, the definition of first-order stable models with extensional predicates can be reduced to the definition proposed in this note. We use this definition to generalize the results on first-order splitting from Ferraris et al. (2009). In particular, we look at the splitting lemma from Ferraris et al. (2009), which showed that under certain conditions the stable models of a formula can be computed by computing the stable models of the same formula with respect to smaller sets of intensional predicates. We find that a straightforward infinitary counterpart to the splitting lemma does not hold, and show how the lemma needs to be modified for the infinitary case. The situation is similar for the splitting theorem discussed above. The infinitary splitting theorem is used to generalize the lemma on explicit definitions due to Ferraris (2005), which describes how adding explicit definitions to a program affects its stable models. In the version presented in this note, the program can include infinitary formulas and the definition can be recursive.

## 2 Review: Infinitary Formulas and their Stable Models

This review follows Truszczynski (2012), Harrison et al. (2015). Let $\sigma$ be a propositional signature, that is, a set of propositional atoms. For every nonnegative integer $r$, *(infinitary propositional) formulas (over $\sigma$) of rank $r$* are defined recursively, as follows:

- every atom from $\sigma$ is a formula of rank 0,
- if $\mathcal{H}$ is a set of formulas, and $r$ is the smallest nonnegative integer that is greater than the ranks of all elements of $\mathcal{H}$, then $\mathcal{H}^\wedge$ and $\mathcal{H}^\vee$ are formulas of rank $r$,

- if $F$ and $G$ are formulas, and $r$ is the smallest nonnegative integer that is greater than the ranks of $F$ and $G$, then $F \to G$ is a formula of rank $r$.

We will write $\{F, G\}^\wedge$ as $F \wedge G$, and $\{F, G\}^\vee$ as $F \vee G$. The symbols $\top$ and $\bot$ will be understood as abbreviations for $\emptyset^\wedge$ and $\emptyset^\vee$ respectively; $\neg F$ stands for $F \to \bot$, and $F \leftrightarrow G$ stands for $(F \to G) \wedge (G \to F)$. These conventions allow us to view finite propositional formulas over $\sigma$ as a special case of infinitary formulas.

A set or family of formulas is *bounded* if the ranks of its members are bounded from above. For any bounded family $(F_\alpha)_{\alpha \in A}$ of formulas, the formula $\{F_\alpha : \alpha \in A\}^\wedge$ will be denoted by $\bigwedge_{\alpha \in A} F_\alpha$, and similarly for disjunctions.

Subsets of a signature $\sigma$ will be also called *interpretations* of $\sigma$. The satisfaction relation between an interpretation and a formula is defined recursively, as follows:

- For every atom $p$ from $\sigma$, $I \models p$ if $p \in I$.
- $I \models \mathcal{H}^\wedge$ if for every formula $F$ in $\mathcal{H}$, $I \models F$.
- $I \models \mathcal{H}^\vee$ if there is a formula $F$ in $\mathcal{H}$ such that $I \models F$.
- $I \models F \to G$ if $I \not\models F$ or $I \models G$.

An infinitary formula is *tautological* if it is satisfied by all interpretations. Two infinitary formulas are *equivalent* if they are satisfied by the same interpretations.

The *reduct* $F^I$ of a formula $F$ w.r.t. an interpretation $I$ is defined recursively, as follows:

- For every atom $p$ from $\sigma$, $p^I$ is $p$ if $p \in I$, and $\bot$ otherwise.
- $(\mathcal{H}^\wedge)^I$ is $\{G^I \mid G \in \mathcal{H}\}^\wedge$.
- $(\mathcal{H}^\vee)^I$ is $\{G^I \mid G \in \mathcal{H}\}^\vee$.
- $(G \to H)^I$ is $G^I \to H^I$ if $I \models G \to H$, and $\bot$ otherwise.

If $\mathcal{H}$ is a set of infinitary formulas then the *reduct* $\mathcal{H}^I$ is the set $\{F^I : F \in \mathcal{H}\}$. An interpretation $I$ is a *stable model* of a set $\mathcal{H}$ of formulas if it is minimal w.r.t. set inclusion among the interpretations satisfying the reduct $\mathcal{H}^I$.

*Example*
It is clear that $\{q\}$ is the only stable model of (3). Indeed, the reduct of (3) w.r.t. $\{q\}$ is

$$\top \to q, \tag{4}$$

and $\{q\}$ is a minimal model of this formula w.r.t. set inclusion. It is easy to see that (3) has no other stable models.

## 3 $\mathcal{A}$-stable Models

Following Ferraris et al. (2011), we will assume that some atoms in a program are designated “intensional” while all others are regarded as “extensional”.

Recall that $\sigma$ denotes a propositional signature. Let $\mathcal{A} \subseteq \sigma$ be a (possibly infinite) set of atoms. The partial order $\leq_\mathcal{A}$ is defined as follows: for any sets $I, J \subseteq \sigma$, we say that $I \leq_\mathcal{A} J$ if $I \subseteq J$ and $J \setminus I \subseteq \mathcal{A}$. (Intuitively, if the atoms in $\mathcal{A}$ are treated as intensional and all other atoms from $\sigma$ are treated as extensional, the relation holds if $I \subseteq J$ and $I, J$ agree on all extensional atoms.) An interpretation $I$ is called an (infinitary) $\mathcal{A}$-*stable model* of a formula $F$ if it is a minimal model of $F^I$ w.r.t. $\leq_\mathcal{A}$.

Observe that if $\mathcal{A} = \sigma$ then $\mathcal{A}$-stable models of a formula $F$ are the same as stable models. If $\mathcal{A} = \emptyset$ then $\mathcal{A}$-stable models are all models of $F$. Truszczynski observed that an interpretation $I$ satisfies $F$ iff $I$ satisfies $F^I$ (Truszczynski, 2012, Proposition 1). It follows that all $\mathcal{A}$-stable models of $F$ also satisfy $F$.

*Example (continued)*
To illustrate the definition of $\mathcal{A}$-stability, let's find all $\{q\}$-stable models[2] of (3). The stable model $\{q\}$ of (3) is $\{q\}$-stable as well, because it is a minimal model of (4) w.r.t. $\leq_{\{q\}}$. On the other hand, any non-empty set $\mathcal{P}$ of atoms of the form $p(t)$ is $\{q\}$-stable as well. Indeed, the reduct of (3) w.r.t. such a set is an implication whose antecedent has $\bot$ as one of its conjunctive terms. This formula is tautological so that it is satisfied by $\mathcal{P}$. Furthermore, $\mathcal{P}$ is a minimal model w.r.t. $\leq_{\{q\}}$ since any subset of $\mathcal{P}$ will disagree with it on extensional atoms.

The fact that all stable models of (3) are also $q$-stable is an instance of a more general fact: If $I$ is an $\mathcal{A}$-stable model of $F$ and $\mathcal{B}$ is a subset of $\mathcal{A}$ then $I$ is also a $\mathcal{B}$-stable model of $F$. This follows directly from the definition of $\mathcal{A}$-stability.

It is easy to check that $\mathcal{A}$-stability can be defined in two other ways:

*Proposition 1*
The following three conditions are equivalent:

- $I$ is an $\mathcal{A}$-stable model of $F$;
- $I$ is a minimal model (w.r.t. set inclusion) of

$$F^I \wedge \bigwedge_{p \in I \setminus \mathcal{A}} p; \tag{5}$$

- $I$ is a stable model of

$$F \;\wedge \bigwedge_{p \in \sigma \setminus \mathcal{A}} (p \vee \neg p).$$

## 4 Relating Infinitary and First-Order $\mathcal{A}$-Stable Models

As mentioned in the introduction, Truszczynski (2012) showed that infinitary stable models can be viewed as a generalization of first-order stable models in the sense of Ferraris et al. (2011). In this section, we will show that the corresponding result holds for **p**-stable models as well.[3] First, we review Truszczynski's results.

Let $\Sigma$ be a first-order signature, and $I$ be an interpretation of $\Sigma$ with non-empty domain $|I|$. For each element $u$ of $|I|$, by $u^*$ we denote a new object constant, called the *name of* $u$. By $\Sigma^{|I|}$ we denote the signature obtained by adding the names of all elements of $|I|$ to $\Sigma$. An interpretation $I$ is identified with its extension $I'$ to $\Sigma^{|I|}$ in which for each $u$ in $|I|$, $I'(u^*) = u$. By $A_{\Sigma,I}$ we denote the set of all atomic sentences over $\Sigma^{|I|}$ built with relation symbols from $\Sigma$ and names of elements in $|I|$, and by $I^r$ we denote the subset of $A_{\Sigma,I}$ that describes in the obvious way the extents of the relations in $I$. Let $F$ be a formula over signature $\Sigma^{|I|}$. Then the *grounding of* $F$ *w.r.t.* $I$, $gr_I(F)$ is defined recursively, as follows:

[2] Here, we understand $\sigma$ as implicitly defined to be the set containing $q$ and all atoms of the form $p(t)$ where $t$ is a ground term.
[3] The definition of **p**-stable models, where **p** is a list of distinct predicate symbols, can be found in Ferraris et al. (2011), Section 2.3.

- $\mathrm{gr}_I(\bot)$ is $\bot$;
- $\mathrm{gr}_I(p(t_1,\dots,t_k))$ is $p((t_1^I)^*,\dots,(t_k^I)^*)$;
- $\mathrm{gr}_I(t_1=t_2)$ is $\top$ if $t_1^I=t_2^I$ and $\bot$ otherwise;
- $\mathrm{gr}_I(F\odot G)$ is $\mathrm{gr}_I(F)\odot$ $\mathrm{gr}_I(G)$, where $\odot\in\{\wedge,\vee,\rightarrow\}$;
- $\mathrm{gr}_I(\forall xF(x))$ is $\{\mathrm{gr}_I(F^x_{u^*})|u\in|I|\}^\wedge$;
- $\mathrm{gr}_I(\exists xF(x))$ is $\{\mathrm{gr}_I(F^x_{u^*})|u\in|I|\}^\vee$.

(By $F^x_{u^*}$ we denote the result of substituting $u^*$ for all free occurrences of $x$ in $F$.) It is clear that for any first-order sentence $F$ over signature $\Sigma$, $\mathrm{gr}_I(F)$ is an infinitary formula over the signature $A_{\Sigma,I}$.

*Example (continued)*
If $\Sigma$ consists of the unary predicate $p$ and the propositional symbol $q$, and $I$ is an interpretation of $\Sigma$ such that the domain $|I|$ is the set of all ground terms $t$, then the grounding of (2) w.r.t. $I$ is (3). (To simplify notation we identify the name of each term $t$ with $t$.)

According to Theorem 5 from Truszczynski (2012), if $F$ is a first-order sentence and $I$ is an interpretation, then $I$ is a first-order stable model of $F$ iff $I^r$ is an infinitary stable model of $\mathrm{gr}_I(F)$. The theorem below generalizes this result to the case of **p**-stable models. By $\mathbf{p}^I$ we denote the atomic formulas in $A_{\Sigma,I}$ built with predicates from **p**.

*Example (continued)*
If **p** is $p$ then $\mathbf{p}^I$ is the set of all atoms of the form $p(t)$.

*Proposition 2*
For any first-order sentence $F$ over $\Sigma$ and any tuple **p** of distinct predicate symbols from $\Sigma$, an interpretation $I$ is a **p**-stable model of $F$ iff $I^r$ is a $\mathbf{p}^{|I|}$-stable model of $\mathrm{gr}_I(F)$.

*Example (continued)*
Let $I$ be the interpretation that interprets $p$ as identically false and assigns the value $\top$ to $q$. Then $I^r$ is $\{q\}$. Let $J$ be an interpretation that satisfies at least one atomic formula $p(t)$ and assigns the value $\bot$ to $q$. Then $J^r$ is $\{p(t)\mid J\models p(t)\}$ (the same as $\mathcal{P}$ from the previous section). We saw in the previous section that $\{q\}$-stable models of (3) are $\{q\}$ and any non-empty set of atoms of the form $p(t)$. In accordance with the theorem above, $I$ and $J$ are $q$-stable models of (2).

*Proof of Proposition 2*
Consider a first-order sentence $F$ and list of distinct predicate symbols **p**. Let $\mathcal{Q}$ be the set of all predicates occurring in $F$ but not in **p**. Consider an interpretation $I$ of the signature of $F$. By Theorem 2 from Ferraris et al. (2011), $I$ is a **p**-stable model of $F$ iff it is a stable model of

$$F\wedge\bigwedge_{q\in\mathcal{Q}}\forall\mathbf{x}(q(\mathbf{x})\vee\neg q(\mathbf{x})),$$

where $\mathbf{x}$ is a list of distinct object variables the same length as the arity of $q$. By Theorem 5 from Truszczynski (2012), $I$ is a stable model of the formula above iff $I^r$ is a stable model of the grounding of this formula w.r.t. $I$. The grounding of the formula above w.r.t. $I$ is

$$gr_I(F)\wedge\bigwedge_{\substack{q\in\mathcal{Q}\\A\in q^I}}(A\vee\neg A)\,.$$

By Proposition 1, $I^r$ is a stable model of this formula iff it is a $\mathbf{p}^{|I|}$-stable model of $\mathrm{gr}_I(F)$. □

## 5 Review: First-Order Splitting Lemma

The lemma presented in the next section of this note is a generalization of the splitting lemma from Ferraris et al. (2009).

In order to state that lemma, we first review the definition of the predicate dependency graph given in that paper. We say that an occurrence of a predicate symbol or a subformula in a first-order formula $F$ is *positive* if it occurs in the antecedent of an even number of implications and *strictly positive* if it occurs in the antecedent of no implications. An occurrence of a predicate constant is said to be *negated* if it belongs to a subformula of the form $\neg F$, and *nonnegated* otherwise. A *rule* of a first-order formula $F$ is a strictly positive occurrence of an implication in $F$. The *(positive) predicate dependency graph* of a first-order formula $F$ w.r.t. a list $\mathbf{p}$ of distinct predicates, denoted $\mathrm{DG}_{\mathbf{p}}[F]$ is the directed graph that

- has all predicate symbols in $\mathbf{p}$ as its vertices, and
- has an edge from $p$ to $q$ if, for some rule $G \to H$ of $F$,
  - — $p$ has a strictly positive occurrence in $H$, and
  - — $q$ has a positive nonnegated occurrence in $G$.

We say that a partition[4] $\{\mathbf{p}_1, \mathbf{p}_2\}$ of the vertices in a graph $G$ is *separable (on G)* if every strongly connected component of $G$ is a subset of either $\mathbf{p}_1$ or $\mathbf{p}_2$. (Here, we identify the list $\mathbf{p}$ with the set of its members.)

The following assertion is a reformulation of Version 1 of the splitting lemma from Ferraris et al. (2009).

*Splitting Lemma*
If $F$ is a first-order sentence and $\mathbf{p}_1, \mathbf{p}_2$ are lists of distinct predicate symbols such that the partition $\{\mathbf{p}_1, \mathbf{p}_2\}$ is separable on $\mathrm{DG}_{\mathbf{p}_1\mathbf{p}_2}[F]$ then $I$ is a $\mathbf{p}_1\mathbf{p}_2$-stable model of $F$ iff it is both a $\mathbf{p}_1$-stable model and a $\mathbf{p}_2$-stable model of $F$.

## 6 Infinitary Splitting Lemma

The statement of the infinitary splitting lemma refers to the positive dependency graph of an infinitary formula. As we will see, the vertices of this graph correspond to intensional atoms. This definition is similar to the definition of a predicate dependency graph given in Ferraris (2007) and Ferraris et al. (2009) and reviewed in the previous section. The concepts necessary to define the dependency graph of an infinitary formula are all straightforward extensions of the concepts used in the previous section to define the predicate dependency graph in the first-order case. However, because infinitary formulas are not syntactic structures, we have to define these concepts recursively.

We define the set of *strictly positive atoms* of an infinitary formula $F$, denoted $\mathrm{P}(F)$, recursively, as follows:

[4] We understand a partition of $X$ to be a set of disjoint subsets (possibly empty) that cover $X$.

- For every atom $p \in \sigma$, $\mathrm{P}(p)$ is $\{p\}$;
- $\mathrm{P}(\mathcal{H}^\wedge)$ is $\bigcup_{H \in \mathcal{H}} \mathrm{P}(H)$, and so is $\mathrm{P}(\mathcal{H}^\vee)$;
- $\mathrm{P}(G \to H)$ is $\mathrm{P}(H)$.

The set of *positive nonnegated atoms* and the set of *negative nonnegated atoms* of an infinitary formula $F$, denoted $\mathrm{Pnn}(F)$ and $\mathrm{Nnn}(F)$ respectively, were introduced in Lifschitz and Yang (2012). These sets are defined recursively as well:

- For every atom $p \in \sigma$, $\mathrm{Pnn}(p)$ is $\{p\}$;
- $\mathrm{Pnn}(\mathcal{H}^\wedge)$ is $\bigcup_{H \in \mathcal{H}} \mathrm{Pnn}(H)$, and so is $\mathrm{Pnn}(\mathcal{H}^\vee)$;
- $\mathrm{Pnn}(G \to H)$ is $\emptyset$ if $H$ is $\bot$ and $\mathrm{Nnn}(G) \cup \mathrm{Pnn}(H)$ otherwise.

and

- For every atom $p \in \sigma$, $\mathrm{Nnn}(p)$ is $\emptyset$;
- $\mathrm{Nnn}(\mathcal{H}^\wedge)$ is $\bigcup_{H \in \mathcal{H}} \mathrm{Nnn}(H)$, and so is $\mathrm{Nnn}(\mathcal{H}^\vee)$;
- $\mathrm{Nnn}(G \to H)$ is $\emptyset$ if $H$ is $\bot$ and $\mathrm{Pnn}(G) \cup \mathrm{Nnn}(H)$ otherwise.

The set of *rules* of an infinitary formula is defined as follows:

- The rules of $G \to H$ are $G \to H$ and all rules of $H$;
- The rules of $\mathcal{H}^\wedge$ and $\mathcal{H}^\vee$ are the rules of all formulas in $\mathcal{H}$.

*Example (continued)*
The set of positive nonnegated atoms in formula (3) is the same as the set of strictly positive atoms: $\{q\}$. The only rule of formula (3) is the formula itself.

For any infinitary formula $F$ the *(positive) dependency graph* of $F$ (relative to a set of atoms $\mathcal{A}$), denoted $\mathrm{DG}_{\mathcal{A}}[F]$, is the directed graph, that

- has all atoms in $\mathcal{A}$ as its vertices, and
- has an edge from $p$ to $q$ if, for some rule $G \to H$ of $F$,
  - — $p$ is an element of $\mathrm{P}(H)$, and
  - — $q$ is an element of $\mathrm{Pnn}(G)$.

The following statement appears to be a plausible counterpart to the splitting lemma reproduced in Section 5 for infinitary formulas:

If $F$ is an infinitary formula and $\mathcal{P}_1, \mathcal{P}_2$ are sets of atoms such that the partition $\{\mathcal{P}_1, \mathcal{P}_2\}$ is separable on $\mathrm{DG}_{\mathcal{P}_1 \cup \mathcal{P}_2}[F]$ then $I$ is a $\mathcal{P}_1 \cup \mathcal{P}_2$-stable model of $F$ iff it is both a $\mathcal{P}_1$-stable model and a $\mathcal{P}_2$-stable model of $F$. $(*)$

But it turns out that this statement does not hold; in the case of infinitary formulas separability is not a sufficient condition to ensure splittability. Let $F$ be the infinitary conjunction

$$\bigwedge_n (p_{n+1} \to p_n)\,,$$

where the conjunction extends over all integers $n$. Let $\mathcal{P}$ be the set of all atoms $p_n$. Let $\mathcal{P}_1$ be the set $\{p_n \mid n \text{ is even}\}$, and $\mathcal{P}_2$ be the set $\{p_n \mid n \text{ is odd}\}$. Then the partition $\{\mathcal{P}_1, \mathcal{P}_2\}$ is separable on $\mathrm{DG}_{\mathcal{P}}[F]$ (shown in Figure 1). Indeed, the strongly connected components of this graph are singletons. If $I$ is the set of all atoms $p_n$ then the reduct of $F$ w.r.t. $I$

$$\cdots \longrightarrow p_{-1} \longrightarrow p_0 \longrightarrow p_1 \longrightarrow \cdots$$

Fig. 1: Any partition of the vertices in this graph is separable.

is $F$ itself. It is easy to check that $I$ is a $\mathcal{P}_1$-stable model as well as a $\mathcal{P}_2$-stable model of $F$, but is not $\mathcal{P}$-stable. This counterexample shows that $(*)$ is incorrect.

In order to extend the splitting lemma to infinitary formulas, we will need a stronger notion of separability. An *infinite walk* $W$ of a directed graph $G$ is an infinite sequence $(v_1, v_2, \dots)$ of vertices occurring in $G$, such that each pair $v_i, v_{i+1}$ in $W$ corresponds to an edge in $G$. A partition $\{\mathcal{P}_1, \mathcal{P}_2\}$ of the vertices in $G$ will be called *infinitely separable (on G)* if every infinite walk $(v_1, v_2, \dots)$ of $G$ visits either $\mathcal{P}_1$ or $\mathcal{P}_2$ finitely many times, that is either $\{i : v_i \in \mathcal{P}_1\}$ or $\{i : v_i \in \mathcal{P}_2\}$ is finite.

*Proposition 3*
For any graph $G$,

(i) every infinitary separable partition of $G$ is separable, and
(ii) if $G$ has finitely many strongly connected components and $G$ is separable then it is infinitely separable.

*Proof*
(i) We will prove the contrapositive: if $\{\mathcal{P}_1, \mathcal{P}_2\}$ is a partition that is not separable on $G$, then there is some strongly connected component of $G$ that contains at least one vertex from $\mathcal{P}_1$ and at least one vertex from $\mathcal{P}_2$. Let's call these vertices $v$ and $w$, respectively. Since $v$ and $w$ are in the same strongly connected component, each vertex is reachable from the other. Then there is an infinite walk that visits each of these vertices (and therefore both $\mathcal{P}_1$ and $\mathcal{P}_2$) infinitely many times, so that the partition is not infinitely separable on $G$.

(ii) Again we prove the contrapositive: if $\{\mathcal{P}_1, \mathcal{P}_2\}$ is a partition that is not infinitely separable on $G$, then there is some infinite walk $(v_1, v_2, \dots)$ of $G$ that visits both $\mathcal{P}_1$ and $\mathcal{P}_2$ infinitely many times. Since there are only finitely many strongly connected components in $G$, at least one strongly connected component of $\mathcal{P}_1$ and at least one strongly connected component of $\mathcal{P}_2$ must be visited infinitely many times. Call these strongly connected components $C_1$ and $C_2$ respectively; then $C_1$ must be reachable from $C_2$ and vice versa. Then $C_1 = C_2$ so that the partition is not separable on $G$. □

Claim $(*)$ will become correct if we require the partition $\{\mathcal{P}_1, \mathcal{P}_2\}$ to be infinitely separable:

*Infinitary Splitting Lemma*
If $F$ is an infinitary formula and $\mathcal{P}_1, \mathcal{P}_2$ are sets of atoms such that the partition $\{\mathcal{P}_1, \mathcal{P}_2\}$ is infinitely separable on $\mathrm{DG}_{\mathcal{P}_1 \cup \mathcal{P}_2}[F]$ then $I$ is a $\mathcal{P}_1 \cup \mathcal{P}_2$-stable model of $F$ iff it is both a $\mathcal{P}_1$-stable model and a $\mathcal{P}_2$-stable model of $F$.

The splitting lemma reproduced in Section 5 is a consequence of the infinitary splitting lemma in view of Theorem 2 and the following fact:

*Proposition 4*
For any first-order sentence $F$ and tuple $\mathbf{p}$ of distinct predicate symbols, if $\{\mathbf{p}_1, \mathbf{p}_2\}$ is a partition of $\mathbf{p}$ that is separable on $\mathrm{DG}_{\mathbf{p}}[F]$, then for any interpretation $I$, $\{\mathbf{p}_1^I, \mathbf{p}_2^I\}$ is infinitely separable on $\mathrm{DG}_{\mathbf{p}^I}[gr_I(F)]$.

*Proof*
If $\{\mathbf{p}_1, \mathbf{p}_2\}$ is a partition of $\mathbf{p}$ that is separable on $\mathrm{DG}_{\mathbf{p}}[F]$, then for any interpretation $I$, the partition $\{\mathbf{p}_1^I, \mathbf{p}_2^I\}$ is separable on the atomic dependency graph of $\mathrm{gr}_I(F)$ with respect to $\mathbf{p}^I$. Furthermore, it is easy to see that $\mathrm{DG}_{\mathbf{p}^I}[gr_I(F)]$ must have finitely many strongly connected components, so that $\{\mathbf{p}_1^I, \mathbf{p}_2^I\}$ must be infinitely separable on it. □

## 7 Proof of the Infinitary Splitting Lemma

The following two lemmas can be easily proved by induction on the rank of $F$.

*Lemma 1*
If $I$ does not satisfy $F$ then the reduct $F^I$ is equivalent to $\bot$.

*Lemma 2*
If the set $\mathcal{A}$ is disjoint from $\mathrm{P}(F)$ and $I$ satisfies $F$, then $I \setminus \mathcal{A}$ satisfies $F^I$.

In particular, if $I$ satisfies $F$ then $I$ satisfies $F^I$. (This is the direction left-to-right of Proposition 1 from Truszczynski (2012).)

Lemmas 3–5 are similar to Lemmas 3–5 from Ferraris et al. (2009).

*Lemma 3*
For any disjoint sets of atoms $\mathcal{B}_1, \mathcal{B}_2$, interpretation $I$, and formula $F$,

(i) If $\mathcal{B}_2$ is disjoint from $\mathrm{Pnn}(F)$ and $I \setminus \mathcal{B}_1$ satisfies $F^I$ then $I \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$ satisfies $F^I$.
(ii) If $\mathcal{B}_2$ is disjoint from $\mathrm{Nnn}(F)$ and $I \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$ satisfies $F^I$ then $I \setminus \mathcal{B}_1$ satisfies $F^I$.

*Proof*
Both parts of the lemma are proved simultaneously by induction on the rank of $F$. Here, we show only the most interesting case when $F$ is of the form $G \to H$. (i) If $I$ does not satisfy $F$ the reduct is equivalent to $\bot$ so that the proposition is trivially true. Assume that $I \setminus \mathcal{B}_1$ satisfies $G^I \to H^I$ and that $\mathcal{B}_2$ is disjoint from $\mathrm{Pnn}(G \to H)$. Then either $H$ is $\bot$ or $\mathcal{B}_2$ is disjoint from both $\mathrm{Nnn}(G)$ and $\mathrm{Pnn}(H)$. If $H$ is $\bot$ then the set $\mathrm{P}(F)$ is empty, so that $(\mathcal{B}_1 \cup \mathcal{B}_2)$ is disjoint from it. Then by Lemma 2, if $I$ satisfies $F$ then $I \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$ satisfies $F^I$. If, on the other hand, $\mathcal{B}_2$ is disjoint from both $\mathrm{Nnn}(G)$ and $\mathrm{Pnn}(H)$, then by part (i) of the induction hypothesis we may conclude that

$$\text{if } I \setminus \mathcal{B}_1 \text{ satisfies } H^I \text{ then so does } I \setminus (\mathcal{B}_1 \cup \mathcal{B}_2), \tag{6}$$

and by part (ii) of the induction hypothesis we may conclude that

$$\text{if } I \setminus (\mathcal{B}_1 \cup \mathcal{B}_2) \text{ satisfies } G^I \text{ then so does } I \setminus \mathcal{B}_1. \tag{7}$$

Assume that $I \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$ satisfies $G^I$. Then by (7), $I \setminus \mathcal{B}_1$ satisfies $G^I$. Then, since $I \setminus \mathcal{B}_1$ satisfies $G^I \to H^I$, that interpretation must satisfy $H^I$. Then by (6) we can conclude that $I \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$ satisfies $H^I$. It follows that that $I \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$ satisfies $G^I \to H^I$. (ii) Similar to Part (i). □

*Lemma 4*
Let $\mathcal{B}, \mathcal{C}$ be disjoint sets of atoms and let $F$ be an infinitary formula such that there are no edges from $\mathcal{B}$ to $\mathcal{C}$ in $\mathrm{DG}_{\mathcal{B}\cup\mathcal{C}}[F]$. If $I \setminus (\mathcal{B} \cup \mathcal{C})$ satisfies $F^I$ then so does $I \setminus \mathcal{B}$.

*Proof*
The proof is by induction on the rank of $F$. Again we show only the most interesting case when $F$ is of the form $G \rightarrow H$. Assume that $I\setminus(\mathcal{B}\cup\mathcal{C})$ satisfies $(G \rightarrow H)^I = G^I \rightarrow H^I$. We need to show that $I \setminus \mathcal{B}$ also satisfies $G^I \rightarrow H^I$. If $\mathcal{B}$ is disjoint from $\mathrm{P}(H)$, then by Lemma 2, $I \setminus \mathcal{B}$ satisfies $H^I$, and therefore satisfies $G^I \rightarrow H^I$. If, on the other hand, $\mathcal{B}$ is not disjoint from $\mathrm{P}(H)$ then $\mathcal{C}$ must be disjoint from $\mathrm{Pnn}(G)$, because there are no edges from $\mathcal{B}$ to $\mathcal{C}$ in $\mathrm{DG}_{\mathcal{B}\cup\mathcal{C}}[G \rightarrow H]$. Then by Lemma 3(i), $I \setminus (\mathcal{B}\cup\mathcal{C})$ satisfies $G^I$. Since we assumed that $I \setminus (\mathcal{B} \cup \mathcal{C})$ satisfies $G^I \rightarrow H^I$, it follows that $I \setminus (\mathcal{B} \cup \mathcal{C})$ satisfies $H^I$. Since every edge in $\mathrm{DG}_{\mathcal{B}\cup\mathcal{C}}[H]$ occurs in $\mathrm{DG}_{\mathcal{B}\cup\mathcal{C}}[G \rightarrow H]$ there is no edge from $\mathcal{B}$ to $\mathcal{C}$ in $\mathrm{DG}_{\mathcal{B}\cup\mathcal{C}}[H]$. Then by the induction hypothesis, $I \setminus \mathcal{B}$ satisfies $H^I$ and therefore satisfies $G^I \rightarrow H^I$. □

*Lemma 5*
For any non-empty graph $G$ and any infinitely separable partition $\{\mathcal{A}_1, \mathcal{A}_2\}$ on $G$, there exists a non-empty subset $\mathcal{B}$ of the vertices in $G$ such that

(i) $\mathcal{B}$ is either a subset of $\mathcal{A}_1$ or a subset of $\mathcal{A}_2$, and
(ii) there are no edges from $\mathcal{B}$ to vertices not in $\mathcal{B}$.

*Proof*
Since $\{\mathcal{A}_1, \mathcal{A}_2\}$ is infinitely separable on $G$, there is some vertex $b$ such that the set of vertices reachable from $b$ is either a subset of $\mathcal{A}_1$ or a subset of $\mathcal{A}_2$. (If no such $b$ existed then $\mathcal{A}_1$ would be reachable from every vertex in $\mathcal{A}_2$ and vice versa, and we could construct an infinite walk visiting both elements of the partition infinitely many times.) It is easy to see that the set of all vertices reachable from $b$ satisfies both (i) and (ii). □

*Proof of the Infinitary Splitting Lemma*
Let $F$ be an infinitary formula such that the partition $\{\mathcal{A}_1, \mathcal{A}_2\}$ is infinitely separable on $\mathrm{DG}_{\mathcal{A}_1\cup\mathcal{A}_2}[F]$. We need to show that $I$ is an $\mathcal{A}_1\cup\mathcal{A}_2$-stable model of $F$ iff it is an $\mathcal{A}_1$-stable model and an $\mathcal{A}_2$-stable model of $F$. The direction left-to-right is obvious. To establish the direction right-to-left, assume that $I$ is both an $\mathcal{A}_1$-stable model and an $\mathcal{A}_2$-stable model of $F$. By Proposition 1 it is sufficient to show that $I$ is a minimal model of

$$F^I \wedge \bigwedge_{p \in I\setminus(\mathcal{A}_1\cup\mathcal{A}_2)} p. \tag{8}$$

Clearly, $I$ satisfies this formula. It remains to show that $I$ is minimal. Assume there is some non-empty subset $X$ of $I$ such that $I\setminus X$ satisfies (8). Then $I\setminus X$ satisfies the second conjunctive term of (8), so $I \setminus (\mathcal{A}_1 \cup \mathcal{A}_2) \subseteq I \setminus X$. Consequently, $X \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$. Consider the sets $X \cap \mathcal{A}_1$ and $X \cap \mathcal{A}_2$. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are infinitely separable on $\mathrm{DG}_{\mathcal{A}_1\cup\mathcal{A}_2}[F]$, the sets $X \cap \mathcal{A}_1$ and $X \cap \mathcal{A}_2$ must be infinitely separable on $\mathrm{DG}_X[F]$. Then by Lemma 5, there is some non-empty set $\mathcal{B}$ that is either a subset of $X \cap \mathcal{A}_1$ or a subset of $X \cap \mathcal{A}_2$ and such that there are no edges from $\mathcal{B}$ to $X \setminus \mathcal{B}$. We will show that $I \setminus \mathcal{B}$ satisfies

$$F^I \wedge \bigwedge_{p \in I\setminus\mathcal{A}_1} p, \tag{9}$$

which contradicts the assumption that $I$ is an $\mathcal{A}_1$-stable model of $F$. Since $I \setminus X$ satisfies the first conjunctive term of (9), by Lemma 4 so does $I \setminus \mathcal{B}$. Assume, for instance, that $\mathcal{B}$ is a subset of $X \cap \mathcal{A}_1$. Then $\mathcal{B}$ is a subset of $\mathcal{A}_1$, so that $I \setminus \mathcal{A}_1$ is a subset of $I \setminus \mathcal{B}$. We may conclude that $I \setminus B$ satisfies the second conjunction term of (9) as well. $\square$

## 8 Infinitary Splitting Theorem

The infinitary splitting lemma can be used to prove the following theorem, which is similar to the splitting theorem from Ferraris et al. (2009).

*Infinitary Splitting Theorem*
Let $F, G$ be infinitary formulas, and $\mathcal{A}_1, \mathcal{A}_2$ be disjoint sets of atoms such that the partition $\{\mathcal{A}_1, \mathcal{A}_2\}$ is infinitely separable on $\mathrm{DG}_{\mathcal{A}_1 \cup \mathcal{A}_2}[F \wedge G]$. If $\mathcal{A}_2$ is disjoint from $\mathrm{P}(F)$, and $\mathcal{A}_1$ is disjoint from $\mathrm{P}(G)$, then for any interpretation $I$, $I$ is an $\mathcal{A}_1 \cup \mathcal{A}_2$-stable model of $F \wedge G$ iff it is both an $\mathcal{A}_1$-stable model of $F$ and an $\mathcal{A}_2$-stable model of $G$.

*Example (continued)*
Consider the conjunction of (3) with the formula $\mathcal{P}^\wedge$ where $\mathcal{P}$ is as before some non-empty set of atoms of the form $p(t)$. We saw previously that $\{q\}$ and all non-empty sets of atoms of the form $p(t)$ are $\{q\}$-stable models of (3). It is easy to check that $\sigma \backslash \{q\}$-stable models of $\mathcal{P}^\wedge$ are $\mathcal{P}$ and $\mathcal{P} \cup \{q\}$. In accordance with the splitting theorem, $\mathcal{P}$ is the only stable model of this formula.

The following lemma, analogous to Theorem 3 from Ferraris et al. (2011), is used to prove the infinitary splitting theorem.

*Lemma 6*
For any infinitary formulas $F, G$, if $\mathcal{A}$ is disjoint from $\mathrm{P}(G)$ then $I$ is an $\mathcal{A}$-stable model of $F \wedge G$ iff it is an $\mathcal{A}$-stable model of $F$ and satisfies $G$.

*Proof*
$\Leftarrow$: Assume $I$ is an $\mathcal{A}$-stable model of $F$ and $I$ satisfies $G$. Since $I$ satisfies $G$ it satisfies $G^I$. Since $I$ is an $\mathcal{A}$-stable model of $F$, it is a minimal w.r.t. $\leq_\mathcal{A}$ among the models of $F$, and consequently among the models of $F \wedge G$.

$\Rightarrow$: Assume $I$ is an $\mathcal{A}$-stable model of $F \wedge G$. Then $I$ is a minimal model of $(F \wedge G)^I$ w.r.t. $\leq_\mathcal{A}$. So $I$ satisfies $F \wedge G$ and therefore satisfies $G$. It remains to show that there is no proper subset $J$ of $I$ such that $I \setminus J \subseteq \mathcal{A}$ and $J$ satisfies $F^I$. Assume that there is some such $J$. Then $J$ must not satisfy $G^I$. (If it did, then $I$ would not be minimal with respect to $\leq_\mathcal{A}$ among the models of $(F \wedge G)^I$.) Let $\mathcal{A}'$ denote $I \setminus J$. Since $\mathcal{A}$ is disjoint from $\mathrm{P}(G)$, so is $\mathcal{A}'$. So by Lemma 2, $I \setminus \mathcal{A}' = J$ must satisfy $G^I$. Contradiction. $\square$

*Proof of the Infinitary Splitting Theorem*
Let $F, G$ be infinitary formulas and let $\mathcal{A}_1, \mathcal{A}_2$ be disjoint sets of atoms such that the partition $\{\mathcal{A}_1, \mathcal{A}_2\}$ is infinitely separable on $\mathrm{DG}_{\mathcal{A}_1 \cup \mathcal{A}_2}[F \wedge G]$ and conditions (i–ii) of the infinitary splitting theorem hold. By the infinitary splitting lemma, $I$ is an $\mathcal{A}_1 \cup \mathcal{A}_2$-stable model of $F \wedge G$ iff it is both an $\mathcal{A}_1$-stable model and an $\mathcal{A}_2$-stable model of $F \wedge G$. Since $\mathcal{A}_2$ is disjoint from $\mathrm{P}(F)$, by Lemma 6, $I$ is an $\mathcal{A}_2$-stable model of $F \wedge G$ iff it is an $\mathcal{A}_2$-stable model of $G$ and it satisfies $F$. Similarly, $I$ is an $\mathcal{A}_1$-stable model of $F \wedge G$ iff it is

an $\mathcal{A}_1$-stable model of $F$ and it satisfies $G$. It remains to observe that if $I$ is an $\mathcal{A}_2$-stable model of $F$ then it satisfies $F$, and similarly if $I$ is an $\mathcal{A}_1$-stable model of $G$. □

## 9 Application: Infinitary Definitions

About a formula $G$ and a set $\mathcal{Q}$ of atoms we will say that $G$ is a *definition for* $\mathcal{Q}$ if it is a conjunction of a set of formulas of the form $H \wedge \mathcal{C}^\wedge \to q$, where $q$ is an atom in $\mathcal{Q}$, $C$ is a subset of $\mathcal{Q}$ (possibly empty), and no atoms from $\mathcal{Q}$ occur in $H$.[5]

A simple special case is "explicit definitions": conjunctions of formulas $H \to q$ such that atoms from $\mathcal{Q}$ don't occur in any $H$. The conjunction of the formulas

$$p_{\alpha\beta} \to q_{\alpha\beta} \quad \text{and} \quad q_{\alpha\beta} \wedge q_{\beta\gamma} \to q_{\alpha\gamma}$$

for all $\alpha, \beta, \gamma$ from some set of indices, which represents the usual recursive definition of transitive closure, is a definition in our sense as well. On the other hand, the formula $\neg q \to q$ is not a definition.

The following theorem shows that all definitions are "conservative".

*Theorem on Infinitary Definitions*
For any infinitary formula $F$, any set $\mathcal{Q}$ of atoms that do not occur in $F$, and any definition $G$ for $\mathcal{Q}$, the map $I \mapsto I \setminus \mathcal{Q}$ is a 1-1 correspondence between the stable models of $F \wedge G$ and the stable models of $F$.

This theorem generalizes the lemma on explicit definitions due to Ferraris (2005) in two ways: it applies to infinitary formulas, and it allows definitions to be recursive.

*Lemma 7*
If all atoms that occur in $F$ belong to $\mathcal{A}$ then, for any interpretation $I$, $I$ is an $\mathcal{A}$-stable model of $F$ iff $I \cap \mathcal{A}$ is a stable model of $F$.

*Proof*
If all atoms that occur in $F$ belong to $\mathcal{A}$ then

$$F^{I \cap \mathcal{A}} \wedge \bigwedge_{p \in I \setminus (I \cap \mathcal{A})} p$$

is identical to (5). □

*Lemma 8*
Let $G$ be a definition for a set $\mathcal{Q}$ of atoms, and let $I$ be a model of $G$. For any subset $K$ of $I$ such that $K \setminus \mathcal{Q} = I \setminus \mathcal{Q}$, $K$ satisfies $G^I$ iff $K$ satisfies $G$.

*Proof.*
We can show that $K$ satisfies a conjunctive term $H \wedge \mathcal{C}^\wedge \to q$ of $G$ iff $K$ satisfies its

[5] The relation *p occurs in F* is defined recursively in a straightforward way.

reduct $H^I \wedge (\mathcal{C}^\wedge)^I \to q^I$ as follows:

$$
\begin{array}{ll}
\quad K \not\models H^I \wedge (C^\wedge)^I \to q^I & \\
\text{iff } K \models H^I,\ K \models (C^\wedge)^I,\ \text{and } K \not\models q^I & \\
\text{iff } K \models H^I,\ K \models (C^\wedge)^I,\ \text{and } q \notin K & (\text{because } K \subseteq I) \\
\text{iff } I \models H^I,\ K \models (C^\wedge)^I,\ \text{and } q \notin K & (K \text{ and } I \text{ agree on atoms occurring in } H) \\
\text{iff } I \models H,\ K \models (C^\wedge)^I,\ \text{and } q \notin K & \\
\text{iff } K \models H,\ K \models (C^\wedge)^I,\ \text{and } q \notin K & (K \text{ and } I \text{ agree on atoms occurring in } H) \\
\text{iff } K \models H,\ C \subseteq K,\ \text{and } q \notin K & (K \subseteq I) \\
\text{iff } K \not\models H \wedge C^\wedge \to q. \quad \square &
\end{array}
$$

*Lemma 9*
Let $G$ be a definition for a set $\mathcal{Q}$ of atoms. For any set $J$ of atoms disjoint from $\mathcal{Q}$ there exists a unique $\mathcal{Q}$-stable model $I$ of $G$ such that $I \setminus \mathcal{Q} = J$.

*Proof*
Let $I$ be the intersection of all models $K$ of $G$ such that $K \setminus \mathcal{Q} = J$. We will show first that $I$ satisfies $G$. Assume otherwise, and take a conjunctive term $H \wedge C^\wedge \to q$ of $G$ that is not satisfied by $I$. Then $I$ satisfies $H$, $C \subseteq I$, and $q \notin I$. By the choice of $I$, it follows that there is a model $K$ of $G$ such that $K \setminus Q = J$ and $q \notin K$. On the other hand, since $I$ satisfies $H$ and does not differ from $K$ on atoms occurring in $H$, $K$ satisfies $H$. Since $C \subseteq I \subseteq K$, $K$ satisfies $C^\wedge$. Hence $K$ does not satisfy one of the conjunctive terms of $G$, which is a contradiction. Thus $I$ is a model of $G$, and consequently a model of $G^I$. To prove that it is $\mathcal{Q}$-stable, consider any model $K$ of $G^I$ such that $K \leq_{\mathcal{Q}} I$. By Lemma 8, $K$ is also a model $G$. By the choice of $I$, it follows that $I \subseteq K$. Consequently $K = I$.

It remains to show that $I$ is unique. Let $K$ be a $\mathcal{Q}$-stable model of $G$ such that $K \setminus Q = J$. It is easy to see that $I \subseteq K$. Furthermore, $K$ satisfies $G^K$ and $I$ satisfies $G$, so by Lemma 8, $I$ satisfies $G^K$. Since $I \leq_{\mathcal{Q}} K$, it follows that $I = K$. $\square$

*Proof of Theorem on Infinitary Definitions*
Let $\sigma$ denote the set of all atoms occurring in $F \wedge G$. Since atoms from $\mathcal{Q}$ do not occur in $F$ and $\mathrm{P}(G) \subseteq \mathcal{Q}$, there are no edges from $\sigma \setminus \mathcal{Q}$ to $\mathcal{Q}$ in $\mathrm{DG}_\sigma[F \wedge G]$. Consequently the partition $\{\sigma \setminus \mathcal{Q}, \mathcal{Q}\}$ is infinitely separable on this graph. By the splitting theorem for infinitary formulas, an interpretation $I$ is a stable model of $F \wedge G$ iff it is a $(\sigma \setminus \mathcal{Q})$-stable model of $F$ and a $\mathcal{Q}$-stable model of $G$. Consider a stable model $I$ of $F \wedge G$. We have seen that $I$ is a $(\sigma \setminus \mathcal{Q})$-stable model of $F$. By Lemma 7, it follows that $I \setminus \mathcal{Q}$ is a stable model of $F$. Consider now a stable model $J$ of $F$, and let $S$ be the set of all interpretations $I$ such that $J = I \setminus \mathcal{Q}$. We will show that $S$ contains exactly one stable model of $F \wedge G$, or equivalently, that there is exactly one interpretation that is a $(\sigma \setminus \mathcal{Q})$-stable model of $F$ and a $\mathcal{Q}$-stable model of $G$ in $S$. By Lemma 7, any interpretation in $S$ is a $(\sigma \setminus \mathcal{Q})$-stable model of $F$. By Lemma 9, $S$ contains exactly one $\mathcal{Q}$-stable model of $G$. $\square$

## 10 Conclusion

In this note, we defined and studied stable models for infinitary propositional formulas with extensional atoms. The use of extensional atoms facilitates a more modular view of logic programs, as evidenced by the Theorem on Infinitary Definitions. The proof of this theorem relies on the Splitting Theorem, and the proof of that theorem makes critical use of the distinction between intensional and extensional atoms.

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