CS311: Discrete Math for Computer Science, Spring 2015

Additional Exercises, with Solutions

Justify your answers.

1. In this problem, $A = \{1, 2, \ldots, 10\}$, $B = \{10, 11, \ldots, 20\}$, $C = \{2, 4, 6, \ldots 20\}$. Find the cardinalities of the sets

(a) $A \cup C$,

(b) $A \cap C$,

(c) $(A \cup B) \setminus C$,

(d) $(A \cap B) \setminus C$,

(e) $(A \cap B) \times C$.

Solution:

(a) $|A \cup C| = |\{1, 2, \ldots, 9, 10, 12, 14, 16, 18, 20\}| = 15$.

(b) $|A \cap C| = |\{2, 4, 6, 8, 10\}| = 5$.

(c) $|(A \cup B) \setminus C| = |\{1, 3, 5, \ldots, 19\}| = 10$.

(d) $|(A \cap B) \setminus C| = |\emptyset| = 0$.

(e) $|(A \cap B) \times C| = |A \cap B| \cdot |C| = 1 \cdot 10 = 10$.

2. Find the cardinality of the set

$$(\{1, 2, \ldots, 100\} \times \{1, 2, \ldots, 101\}) \setminus (\{1, 2, \ldots, 101\} \times \{1, 2, \ldots, 100\}).$$

Solution: Denote

$$\text{the set } \{1, 2, \ldots, 100\} \times \{1, 2, \ldots, 101\} \text{ by } X,$$

$$\text{the set } \{1, 2, \ldots, 101\} \times \{1, 2, \ldots, 100\} \text{ by } Y.$$

Set X consists of the pairs $\langle m, n\rangle$ such that m is between 1 and 100, and n is between 1 and 101; there are 10,100 such pairs. Such a pair $\langle m, n\rangle$ belongs to Y if n is between 1 and 100; there are 10,000 such pairs. Consequently the cardinality of $X \setminus Y$ is $10,100 - 10,000$, which equals 100.

3. Find sets A and B such that

$$A \setminus B = \{1, 5, 7, 8\},$$
$$B \setminus A = \{2, 10\},$$
$$A \cap B = \{3, 6, 9\}.$$

Answer: $A = \{1, 3, 5, 6, 7, 8, 9\}$, $B = \{2, 3, 6, 9, 10\}$.

4. Can you conclude that $A = B$ if A, B, C are sets such that

(a) $A \cup C = B \cup C$?

Answer: No. Counterexample: $A = \{1, 2, 3\}, B = \{1, 2\}, C = \{3\}$.

(b) $A \cap C = B \cap C$?

Answer: No. Counterexample: $A = \{1, 2, 3\}, B = \{3, 4, 5\}, C = \{3\}$.

5. For any sets A and B, if $|A \times B| = 91$ then at least one of the sets A, B is a singleton. True or false?

Answer: False. Example: $A = \{1, 2, \ldots, 7\}$; $B = \{1, 2, \ldots, 13\}$.

6. Consider the relation $x = 2y + 1$ between real numbers x, y. Is it reflexive? Is it symmetric? Is it transitive?

Solution: Denote the given relation by R, so that

$$xRy \leftrightarrow x = 2y + 1.$$

This relation is not reflexive, because the condition $1R1$ does not hold.

This relation is not symmetric, because the condition $1R3$ holds, but the condition $3R1$ doesn't.

This relation is not transitive, because the conditions $1R3$ and $3R7$ hold, but the condition $1R7$ doesn't.

7. What is the total number of binary relations on the set $\{1, \ldots, 10\}$? How many of them are reflexive?

Solution: A binary relation is an arbitrary subset of the set $\{1, \ldots, 10\} \times \{1, \ldots, 10\}$. So the total number of binary relations is 2^{100}. Such a subset is a reflexive relation if it contains 10 pairs of the form $\langle n, n \rangle$. So the number of reflexive relations is 2^{90}.