CS311: Discrete Math for Computer Science, Spring 2015

Additional Exercises, with Solutions

1. Prove that the assertion

$$m = F_i \wedge n = F_{i+1} \wedge i \leq 10$$

where F_i is the i-th Fibonacci number, is a loop invariant for the loop

$$\begin{array}{l} \textbf{while } i < 10 \textbf{ do} \\ \quad i \leftarrow i + 1; \\ \quad k \leftarrow m + n; \\ \quad m \leftarrow n; \\ \quad n \leftarrow k \\ \textbf{enddo} \end{array}$$

Solution:

$$\begin{array}{l} \{m = F_i \wedge n = F_{i+1} \wedge i \leq 10 \wedge i < 10\} \\ \{m = F_i \wedge n = F_{i+1} \wedge i < 10\} \\ \{n = F_{i+1} \wedge m + n = F_{i+2} \wedge i + 1 \leq 10\} \\ i \leftarrow i + 1; \\ \{n = F_i \wedge m + n = F_{i+1} \wedge i \leq 10\} \\ k \leftarrow m + n; \\ \{n = F_i \wedge k = F_{i+1} \wedge i \leq 10\} \\ m \leftarrow n; \\ \{m = F_i \wedge k = F_{i+1} \wedge i \leq 10\} \\ n \leftarrow k \\ \{m = F_i \wedge n = F_{i+1} \wedge i \leq 10\} \end{array}$$

2. Determine which of the assertions

$$n \geq 10, \quad n \,|\, 10, \quad 10 \,|\, n$$

are loop invariants for the loop

$$\begin{array}{l} \textbf{while } n < 10 \textbf{ do} \\ \quad n \leftarrow n \times 2 \\ \textbf{enddo} \end{array}$$

Solution. Assertion $n \geq 10$ is a loop invariant:

$$\begin{array}{l} \{n \geq 10 \wedge n < 10\} \\ \{\mathit{false}\} \\ \{n \times 2 \geq 10\} \\ n \leftarrow n \times 2 \\ \{n \geq 10\} \end{array}$$

Assertion $n \mid 10$ is not a loop invariant; counterexample: $n = 10$.
Assertion $10 \mid n$ is a loop invariant:

$$\begin{array}{l} \{10 \mid n \land n < 10\} \\ \{10 \mid n\} \\ \{10 \mid n \times 2\} \\ n \leftarrow n \times 2 \\ \{10 \mid n\} \end{array}$$

3. The sequence $A_0, A_1, \ldots$ is defined by the formulas

$$A_n = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{n+1}{2}, & \text{otherwise.} \end{cases}$$

(a) Prove that all members of this sequence are integers.

Solution. Case 1: n is even. Then $A_n = \frac{n}{2}$, which is an integer, Case 2: n is odd. Then $A_n = \frac{n+1}{2}$; since $n+1$ is even, this is an integer,

(b) Prove that for every n, $A_{n+2} = A_n + 1$.

Solution. Case 1: n is even. Then $n+2$ is even too, and

$$A_{n+2} = \frac{n+2}{2} = \frac{n}{2} + 1 = A_n + 1.$$

Case 2: n is odd. Then $n+2$ is odd too, and

$$A_{n+2} = \frac{(n+2)+1}{2} = \frac{n+3}{2} = \frac{n+1}{2} + 1 = A_n + 1.$$

4. The sequence $B_0, B_1, \ldots$ is defined by the formulas

$$\begin{array}{l} B_0 = 0, \\ B_1 = 1, \\ B_{n+2} = 4B_{n+1} - B_n. \end{array}$$

(i) Find an explicit formula for B_n. (ii) Determine how the formula will change if we replace the first two equations by

$$\begin{array}{l} B_0 = 1, \\ B_1 = 2. \end{array}$$

Solution. First we need to find the values of c for which the sequence $1, c, c^2, \ldots$ satisfies the condition $B_{n+2} = 4B_{n+1} - B_n$. If we take $n = 0$ in this formula and replace B_n with c^n, we get the equation $c^2 = 4c - 1$. From this equation we find:

$$c_1 = 2 + \sqrt{3}, \quad c_2 = 2 - \sqrt{3}.$$

It follows that any sequence B_n defined by a formula of the form

$$B_n = a\left(2+\sqrt{3}\right)^n + b\left(2-\sqrt{3}\right)^n$$

satisfies the condition $B_{n+2} = 4B_{n+1} - B_n$. It remains to find the coefficients a and b.

(i) From the initial conditions $B_0 = 0$, $B_1 = 1$ we find:

$$\begin{aligned} a + b &= 0, \\ a\left(2+\sqrt{3}\right) + b\left(2-\sqrt{3}\right) &= 1. \end{aligned}$$

From these equations,

$$a = \frac{1}{2\sqrt{3}}, \quad b = -\frac{1}{2\sqrt{3}}.$$

So the formula for B_n in this case is

$$B_n = \frac{1}{2\sqrt{3}}\left(2+\sqrt{3}\right)^n - \frac{1}{2\sqrt{3}}\left(2-\sqrt{3}\right)^n.$$

(ii) From the initial conditions $B_0 = 1$, $B_1 = 2$ we find:

$$\begin{aligned} a + b &= 1, \\ a\left(2+\sqrt{3}\right) + b\left(2-\sqrt{3}\right) &= 2. \end{aligned}$$

From these equations,

$$a = b = \frac{1}{2}.$$

So the formula for B_n in this case is

$$B_n = \frac{1}{2}\left(2+\sqrt{3}\right)^n + \frac{1}{2}\left(2-\sqrt{3}\right)^n.$$

5. Draw a graph with the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Answer: