CS311: Discrete Math for Computer Science, Spring 2015

Homework Assignment 3, with Solutions

1. The following table shows the first several members of the sequence U_n:

n	1	2	3	4	5	6	...
U_n	2	0	4	0	6	0	...

Give a definition by cases for this sequence.

Answer:

$$U_n = \begin{cases} 0, & \text{if } n \text{ is even,} \\ n+1, & \text{otherwise.} \end{cases}$$

2. Function f is defined by cases:

$$f(x) = \begin{cases} -x-1, & \text{if } x < -1, \\ 0, & \text{if } -1 \le x \le 1, \\ x-1, & \text{if } x > 1. \end{cases}$$

(i) Rewrite the definition of f in logical notation.

Answer:

$$\begin{aligned} &\forall x((x < -1 \to f(x) = -x-1) \\ &\wedge (-1 \le x \le 1 \to f(x) = 0) \\ &\wedge (x > 1 \to f(x) = x-1)). \end{aligned}$$

(ii) Prove that for every real number x, $f(x) \ge 0$.

Solution:

Case 1: $x < -1$. Then $-x > 1$, so that $f(x) = -x - 1 > 1 - 1 = 0$.

Case 2: $-1 \le x \le 1$. Then $f(x) = 0$.

Case 3: $x > 1$. Then $f(x) = x - 1 > 0$.

(iii) We conjecture that there exist coefficients a, b, c such that for all real numbers x

$$f(x) = a|x+1| + b|x-1| + c.$$

Find a triple of numbers a, b, c that may satisfy this condition.

Solution: By substituting -2, 0, and 2 for x, we get the equations

$$\begin{aligned} a + 3b + c &= 1, \\ a + b + c &= 0, \\ 3a + b + c &= 1. \end{aligned}$$

From these equations we find: $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = -1$. So the formula is

$$f(x) = \frac{1}{2}|x+1| + \frac{1}{2}|x-1| - 1.$$

(iv) Check that for the numbers a, b, c that you found this condition is indeed satisfied for all values of x.

Solution:

Case 1: $x < -1$. Then $|x+1| = -x-1$ and $|x-1| = -x+1$, so that

$$f(x) = \frac{1}{2}|x+1| + \frac{1}{2}|x-1| - 1 = \frac{1}{2}(-x-1) + \frac{1}{2}(-x+1) - 1 = -x-1.$$

Case 2: $-1 \le x \le 1$. Then $|x+1| = x+1$ and $|x-1| = -x+1$, so that

$$f(x) = \frac{1}{2}|x+1| + \frac{1}{2}|x-1| - 1 = \frac{1}{2}(x+1) + \frac{1}{2}(-x+1) - 1 = 0.$$

Case 3: $x > 1$. Then $|x+1| = x+1$ and $|x-1| = x-1$, so that

$$f(x) = \frac{1}{2}|x+1| + \frac{1}{2}|x-1| - 1 = \frac{1}{2}(x+1) + \frac{1}{2}(x-1) - 1 = x-1.$$