CS311: Discrete Math for Computer Science, Spring 2015

Homework Assignment 4, with Solutions

1. Find coefficients a, b, c, d such that formula (5) from Part 3 of Lecture Notes is satisfied for all values of n.

Solution: By substituting 0, 1, 2 and 3 for n in the formula

$$S_n = an^3 + bn^2 + cn + d,$$

we get the equations

$$\begin{aligned} 0 &= d, \\ 1 &= a + b + c + d, \\ 5 &= 8a + 4b + 2c + d, \\ 14 &= 27a + 9b + 3c + d. \end{aligned}$$

From these equations we find: $a = \frac{1}{3}$, $b = \frac{1}{2}$, $c = \frac{1}{6}$, $d = 0$.

2. (a) Find a formula for

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)}$$

by examining the values of this expression for small values of n.

Answer: $\frac{n}{n+1}$.

(b) Prove the formula you conjectured in part (a).

Solution: we will prove the formula

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$$

by induction. *Basis:* When $n = 0$, the formula turns into $0 = \frac{0}{1}$. *Induction step:* assuming that the given formula holds for n, we can prove that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} + \frac{1}{(n+1) \cdot (n+2)} = \frac{n+1}{n+2}$$

as follows:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} + \frac{1}{(n+1) \cdot (n+2)} = \frac{n}{n+1} + \frac{1}{(n+1) \cdot (n+2)}$$

$$= \frac{n \cdot (n+2) + 1}{(n+1) \cdot (n+2)} = \frac{n^2 + 2n + 1}{(n+1) \cdot (n+2)} = \frac{(n+1)^2}{(n+1) \cdot (n+2)} = \frac{n+1}{n+2}.$$

3. (a) Find a formula for

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!$$

by examining the values of this expression for small values of n.

Answer: $(n+1)! - 1$.

(b) Prove the formula you conjectured in part (a).

Solution: we will prove the formula

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$$

by induction. *Basis:* When $n = 0$, the formula turns into $0 = 1! - 1$. *Induction step:* assuming that the given formula holds for n, we can prove that

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! + (n+1) \cdot (n+1)! = (n+2)! - 1$$

as follows:

$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! + (n+1) \cdot (n+1)! &= (n+1)! - 1 + (n+1) \cdot (n+1)! \\ &= (n+1)!(n+1+1) - 1 \\ &= (n+2)! - 1. \end{aligned}$$

4. Prove that for every nonnegative integer n

$$\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2.$$

Solution: we will prove the formula by induction. *Basis:* When $n = 0$, the formula turns into $0 = (-1) \cdot 2 + 2$. *Induction step:* assuming that the given formula holds for n, we can prove that

$$\sum_{i=1}^{n+1} i2^i = n2^{n+2} + 2$$

as follows:

$$\sum_{i=1}^{n+1} i2^i = \sum_{i=1}^{n} i2^i + (n+1)2^{n+1} = (n-1)2^{n+1} + 2 + (n+1)2^{n+1} = 2n \cdot 2^{n+1} + 2 = n2^{n+2} + 2.$$