CS311: Discrete Math for Computer Science, Spring 2015

Homework Assignment 6, with Solutions

Justify your answers.

1. Determine which of the two given sequences grows faster. If they grow at the same rate then determine whether they are asymptotically equal.

(a) $(4n^2+1)^3$ and $(8n^3+1)^2$.

Solution. By expanding these expressions we get two polynomials that have the same term with the highest degree of n, namely $64n^6$. It follows that the given sequences are asymptotically equal.

(b) $(\sqrt{n})^n$ and $n^{\sqrt{n}}$.

Solution:

$$\frac{(\sqrt{n})^n}{n^{\sqrt{n}}} = \frac{(n^{\frac{1}{2}})^n}{n^{\sqrt{n}}} = \frac{n^{\frac{n}{2}}}{n^{\sqrt{n}}} = n^{\frac{n}{2}-\sqrt{n}}.$$

This expression goes to infinity as n grows. It follows that $(\sqrt{n})^n$ grows faster.

(c) $n! + (n+1)!$ and $(n+1)!$.

Solution.

$$\frac{n! + (n+1)!}{(n+1)!} = \frac{1}{n+1} + 1.$$

This expression goes to 1 as n grows. It follows that the sequences are asymptotically equal.

(d) H_n and $\log_2 n$.

Solution. The sequence H_n grows at the same rate as $\ln n$. It follows that H_n grows at the same rate as $\log_2 n$, but is not asymptotically equal to it.

(e) F_n and Z_n (these sequences are defined in Part 5 of Lecture Notes).

Solution. The sequence F_n is asymptotically equal to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

The sequence Z_n is asymptotically equal to

$$\frac{25+9\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

It follows that they grow at the same rate, but are not asymptotically equal.

2. Without a calculator, find approximate values of the expressions

(a) $1^4 + 2^4 + \cdots + 20^4$,

(b) $\sqrt[3]{65} + \sqrt[3]{66} + \cdots + \sqrt[3]{512}$.

Solution. In both cases, we use the fact that $\sum_{i=1}^{n} i^k$ is approximately equal to $\frac{1}{k+1} n^{k+1}$ when n is large.

(a) For $k = 4$,

$$1^4 + 2^4 + \cdots + 20^4 \approx \frac{1}{5} \cdot 20^5 = 640000.$$

(b) For $k = \frac{1}{3}$,

$$\begin{aligned}
\sqrt[3]{65} + \sqrt[3]{66} + \cdots + \sqrt[3]{512} &= \sum_{i=1}^{512} i^{1/3} - \sum_{i=1}^{64} i^{1/3} \\
&\approx \frac{3}{4} \cdot 512^{4/3} - \frac{3}{4} \cdot 64^{4/3} \\
&= \frac{3}{4} \cdot \left(\sqrt[3]{512}\right)^4 - \frac{3}{4} \cdot \left(\sqrt[3]{64}\right)^4 \\
&= \frac{3}{4} \cdot 8^4 - \frac{3}{4} \cdot 4^4 \\
&= 2880.
\end{aligned}$$

3. Without a calculator, determine whether the number

$$\frac{1}{1001} + \frac{1}{1002} + \cdots + \frac{1}{2500}$$

is greater than 1.

Solution. We use the fact that $\sum_{i=1}^{n} \frac{1}{i}$ is approximately equal to $\ln n$ when n is large.

$$\frac{1}{1001} + \frac{1}{1002} + \cdots + \frac{1}{2500} = \sum_{i=1}^{2500} \frac{1}{i} - \sum_{i=1}^{1000} \frac{1}{i} \approx \ln 2500 - \ln 1000 = \ln 2.5.$$

Since $2.5 < e$, this number is less than 1.