# CS311: Discrete Math for Computer Science, Spring 2015

## Homework Assignment 9, with Solutions

**1.** Find the total number of functions from $\{1, \ldots, 10\}$ onto $\{1, 2\}$.

*Solution.* The total number of functions from $\{1, \ldots, 10\}$ to $\{1, 2\}$ is $2^{10}$, or 1024. Among these functions, 2 are not onto: the function that maps every element of the domain to 1, and the function that maps every element of the domain to 2. So the total number of functions from $\{1, \ldots, 10\}$ onto $\{1, 2\}$ is 1022.

**2.** Recall that by $\mathbf{S}$ we denote the set of all bit strings:

$$\mathbf{S} = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, \ldots\},$$

and that we introduced the following functions:

- Function $l$ from $\mathbf{S}$ to $\mathbf{N}$: $l(x)$ is the length of $x$.
- Function $e$ from $\mathbf{S}$ to $\mathbf{S}$: $e(x)$ is the string $1x$.
- Function $r$ from $\mathbf{S}$ to $\mathbf{S}$: $r(x)$ is the string $x$ reversed.

Determine which of the following formulas are true.

(i) $l \circ e = l$.

*Solution:* False; for instance, $l(e(1)) = l(11) = 2$ but $l(1) = 1$.

(ii) $l \circ r = l$.

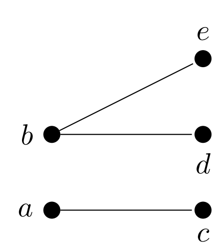
*Solution:* True; $l(r(x))$ is the length of $x$ reversed, which is the same as the length of $x$.

(iii) $e \circ r = r \circ e$.

*Solution:* False; for instance, $e(r(001)) = e(100) = 1100$ but $r(e(001)) = r(1001) = 1001$.

**3.** (a) Draw a bipartite graph with 5 vertices. (b) Find the adjacency matrix of this graph. (c) Determine whether this graph is a tree. (d) How many simple paths are there in this graph?

*Possible solution:*



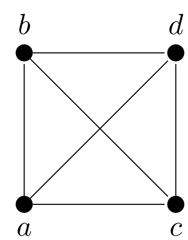

The adjacency matrix is as follows:

| | a | b | c | d | e |
|---|---|---|---|---|---|
| a | 0 | 0 | 1 | 0 | 0 |
| b | 0 | 0 | 0 | 1 | 1 |
| c | 1 | 0 | 0 | 0 | 0 |
| d | 0 | 1 | 0 | 0 | 0 |
| e | 0 | 1 | 0 | 0 | 0 |

This graph is not a tree since it is not connected. There are 5 paths consisting of a single vertex, 6 paths consisting of two vertices, and 2 simple paths consisting of 3 vertices, so that the total number of simple paths is 13.

**4.** How many cycles are there in the complete graph on 4 vertices?

*Solution.* Let the vertices of the graph be $a, b, c, d$:

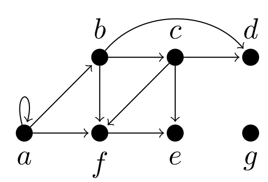



There are 6 cycles that include the vertices $a, b, c$:

$$a, b, c, a; \quad a, c, b, a; \quad b, c, a, b; \quad b, a, c, b; \quad c, a, b, c; \quad c, b, a, c.$$

Similarly, there are 6 cycles including $a, b, d$, 6 cycles including $a, c, d$ and 6 cycles including $b, c, d$. So the total number of cycles including three vertices out of four is 24. There are also 24 cycles including all four vertices: one per each permutation of $a, b, c, d$. So the total number of cycles is 48.

**5.** (a) Find the in-degree and the out-degree of each vertex in the graph shown in the picture. (b) Find the strongly connected components of this graph.




*Answer:*

| Vertex | Indegree | Outdegree |
|---|---|---|
| a | 1 | 3 |
| b | 1 | 3 |
| c | 1 | 3 |
| d | 2 | 0 |
| e | 2 | 0 |
| f | 3 | 1 |
| g | 0 | 0 |

The strongly connected components are $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{e\}$, $\{f\}$, $\{g\}$.