Lecture Notes:
Discrete Mathematics for Computer Science

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Part 3. Triangular Numbers and Their Relatives

Definitions

In the definitions below, n is a nonnegative integer.

The *triangular number* T_n is the sum of all integers from 1 to n:

$$T_n = \sum_{i=1}^{n} i = 1 + 2 + \cdots + n.$$

For instance, $T_4 = 1 + 2 + 3 + 4 = 10$.

By B_n we denote the number of ways to choose two elements out of n. For instance, if we take 5 elements a, b, c, d, e, then there will be 10 ways to choose two:

$$a, b; \quad a, c; \quad a, d; \quad a, e; \quad b, c; \quad b, d; \quad b, e; \quad c, d; \quad c, e; \quad d, e.$$

We conclude that $B_5 = 10$.

By P_n we denote the number of parts into which a plane is divided by n straight lines in general position. ("In general position" means that there are no parallel lines and no multiple intersection points.) For instance, 3 lines in general position divide the plane into 7 parts: a triangle and 6 infinite regions. We conclude that $P_3 = 7$.

By S_n we denote the sum of the squares of all integers from 1 to n:

$$S_n = \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2.$$

For instance, $S_4 = 1^2 + 2^2 + 3^2 + 4^2 = 30$.

By C_n we denote the sum of the cubes of all integers from 1 to n:

$$C_n = \sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3.$$

For instance, $C_4 = 1^3 + 2^3 + 3^3 + 4^3 = 100$.

The *harmonic number* H_n is defined by the formula

$$H_n = \sum_{i=1}^{n} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}.$$

For instance, $H_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$.

The *factorial* of n is the product of all integers from 1 to n:

$$n! = \prod_{i=1}^{n} i = 1 \cdot 2 \cdot \dots \cdot n.$$

For instance, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$. Note that $0! = 1$.

A Formula for Triangular Numbers

Triangular numbers can be calculated using the formula

$$T_n = \frac{n(n+1)}{2}. \tag{1}$$

To prove this formula, we start with two expressions for T_n:

$$\begin{array}{ccccccccccccccc} T_n & = & 1 & + & 2 & + & 3 & + & \cdots & + & (n-2) & + & (n-1) & + & n \\ T_n & = & n & + & (n-1) & + & (n-2) & + & \cdots & + & 3 & + & 2 & + & 1. \end{array}$$

If we add them column by column, we'll get:

$$\begin{aligned} 2T_n &= (n+1) \;+\; (n+1) \;+\; (n+1) \;+\; \cdots \;+\; (n+1) \;+\; (n+1) \;+\; (n+1) \\ &= n(n+1), \end{aligned}$$

and it remains to divide both sides by 2.

There are also other ways to prove formula (1). Consider these identities:

$$\begin{array}{ccccccc} (1+1)^2 & = & 1^2 & + & 2 \cdot 1 \cdot 1 & + & 1^2, \\ (2+1)^2 & = & 2^2 & + & 2 \cdot 2 \cdot 1 & + & 1^2, \\ (3+1)^2 & = & 3^2 & + & 2 \cdot 3 \cdot 1 & + & 1^2, \\ \dots & & & & & & \\ (n+1)^2 & = & n^2 & + & 2 \cdot n \cdot 1 & + & 1^2. \end{array}$$

If we add them column by column, we'll get:

$$2^2 + 3^2 + 4^2 + \cdots + (n+1)^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 + 2T_n + n.$$

Subtract $2^2 + 3^2 + \cdots + n^2$ from both sides:

$$(n+1)^2 = 1^2 + 2T_n + n.$$

Expand the left-hand side and subtract $n+1$ from both sides:

$$n^2 + n = 2T_n.$$

It remains to divide both sides by 2.

A Formula for B_n

To find out how the sequence B_n is related to triangular numbers, we made a table that shows several members of this sequence and several triangular numbers next to each other:

n	B_n	T_n
0	0	0
1	0	1
2	1	3
3	3	6
4	6	10
5	10	15

We see that for $n = 1, \ldots, 5$,

$$B_n = T_{n-1}. \tag{2}$$

Formula (2) holds actually for all positive integers n. This fact can be proved as follows. Number B_n can be thought of as the number of 2-element sets formed from the numbers $1, 2, \ldots, n$. Among these sets, $n-1$ include the number n, because there are $n-1$ choices for the second element of the set—it can be any of the numbers $1, 2, \ldots, n-1$. In addition, there are $n-2$ sets that don't include n but include $n-1$, because there are $n-2$ choices for the second element of the set—it can be any of the numbers $1, 2, \ldots, n-2$. Similarly, there are $n-3$ sets that include neither n nor $n-1$ but include $n-2$, and so on. Finally, there is one set that doesn't include any of the numbers greater than 2; that set consists of 1 and 2. It follows that the total number of 2-element sets formed from the numbers $1, 2, \ldots, n$ is

$$(n-1) + (n-2) + (n-3) + \cdots + 1,$$

which is the triangular number T_{n-1}.

From formulas (1) and (2) we conclude that

$$B_n = \frac{n(n-1)}{2}.$$

A Formula for P_n

To find out how the sequence P_n is related to triangular numbers, we made a table that shows several members of this sequence and several triangular numbers next to each other:

n	P_n	T_n
0	1	0
1	2	1
2	4	3
3	7	6
4	11	10

We see that for $n = 0, \ldots, 4$,

$$P_n = T_n + 1. \tag{3}$$

We will prove later that formula (3) holds actually for all n. For the time being, this claim remains a conjecture.

Using formula (1), we can derive from this conjecture that

$$P_n = \frac{n(n+1)}{2} + 1 = \frac{n^2}{2} + \frac{n}{2} + 1.$$

A Formula for C_n

To find out how the sequence C_n is related to triangular numbers, we made a table that shows several members of this sequence and several triangular numbers next to each other:

n	C_n	T_n
0	0	0
1	1	1
2	9	3
3	36	6
4	100	10

We see that for $n = 0, \ldots, 4$,

$$C_n = (T_n)^2. \tag{4}$$

We will prove later that formula (4) holds actually for all n. For the time being, this claim remains a conjecture.

Using formula (1), we can derive from this conjecture that

$$C_n = \frac{n^2(n+1)^2}{4}.$$

Towards a Formula for S_n

Formulas (1) and (4) show that T_n can be expressed as a quadratic polynomial, and that C_n can be expressed as a polynomial of degree 4:

$$T_n = \frac{1}{2}n^2 + \frac{1}{2}n,$$

$$C_n = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2.$$

This observation leads to the conjecture that S_n can be expressed as a polynomial of degree 3:

$$S_n = an^3 + bn^2 + cn + d. \tag{5}$$

We will see later that such a formula indeed exists. The values of a, b, c, d can be found by the method of method of undetermined coefficients.

How Large are Harmonic Numbers and Factorials?

There are no simple precise formulas for harmonic numbers and factorials. About the sequence H_n we know that it goes to infinity, but very slowly. There is a simple approximate formula for H_n, and we will discuss it later in this course.

It is interesting to compare factorials with the powers of 2:

n	$n!$	2^n
0	1	1
1	1	2
2	2	4
3	6	8
4	24	16
5	120	32

We see that the factorial of n is greater than 2^n when n is 4 or 5, but it is less than or equal to 2^n for smaller values of n. We will prove later that $n! > 2^n$ for all values of n beginning with 4.