CS311: Discrete Math for Computer Science, Spring 2015

Test 2, with Solutions

Open notes. No books, no calculators.

1. Determine whether the given formula is true or false. Justify your answers.

(a) $\forall m \exists n (2 \mid m+n)$. *Solution:* True. Choose $n = -m$. Then $m + n = 0$; 0 is even.

(b) $\exists m \forall n (m-5 \mid n)$. *Solution:* True. Choose $m = 6$. Then $m - 5 = 1$; every integer is a multiple of 1.

2. Function f is defined by the formulas

$$f(x) = \begin{cases} x, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find numbers a, b such that for all values of x

$$f(x) = ax + b|x|.$$

Prove that your formula is correct.

Solution. If $f(x) = ax + b|x|$ for all x then $0 = f(-1) = -a + b$ and $1 = f(1) = a + b$. Solving gives us $a = \frac{1}{2}$ and $b = \frac{1}{2}$. To prove the formula

$$f(x) = \frac{x}{2} + \frac{|x|}{2},$$

consider two cases. Case 1: $x < 0$. Then $|x| = -x$, so that

$$\frac{x}{2} + \frac{|x|}{2} = \frac{1}{2}x - \frac{1}{2}x = 0 = f(x).$$

Case 2: $x \geq 0$. Then $|x| = x$, so that

$$\frac{x}{2} + \frac{|x|}{2} = \frac{1}{2}x + \frac{1}{2}x = x = f(x).$$

3. Prove that for every positive integer n

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$

Solution: We will prove the formula by induction. *Basis:* $n = 1$. The given formula turns into the correct equality $\frac{1}{1 \cdot 3} = \frac{1}{2 \cdot 1 + 1}$. *Induction step:* Assuming that the given formula is true for n, we can prove that

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} + \frac{1}{(2n+1)(2n+3)} = \frac{n+1}{2n+3}$$

as follows:

$$\begin{aligned}\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}+\frac{1}{(2n+1)(2n+3)}&\\ &=\frac{n}{2n+1}+\frac{1}{(2n+1)(2n+3)}\\ &=\frac{n(2n+3)+1}{(2n+1)(2n+3)}\\ &=\frac{2n^2+3n+1}{(2n+1)(2n+3)}\\ &=\frac{(n+1)(2n+1)}{(2n+1)(2n+3)}\\ &=\frac{n+1}{2n+3}.\end{aligned}$$

4. Prove that for every positive integer n, 43 divides $6^{n+1}+7^{2n-1}$.

Solution. We will prove this assertion by induction. *Basis:* $n=1$. Then

$$6^{n+1}+7^{2n-1}=6^2+7^1=43.$$

Induction step: Suppose that 43 divides $6^{n+1}+7^{2n-1}$. To prove that 43 divides $6^{n+2}+7^{2n+1}$, we rewrite this expression as follows:

$$6^{n+2}+7^{2n+1}=6^{n+1}\cdot 6+7^{2n-1}\cdot 49=(6^{n+1}+7^{2n-1})\cdot 6+7^{2n-1}\cdot 43.$$

In the last expression, each of the two summands is a multiple of 43.

5. The numbers $A_0, A_1, A_2, \ldots$ are defined by the formulas

$$\begin{aligned}&A_0=0,\\ &A_{n+1}=n\cdot(A_n+1).\end{aligned}$$

Prove that $A_n<n!$.

Solution. Consider two cases. Case 1: $n=0$. Then $A_n=A_0=0<1=0!=n!$. Case 2: $n>0$. We will prove the inequality $A_n<n!$ by induction. *Basis:* $n=1$. Then $A_n=A_1=0<1=1!=n!$. *Induction step:* Assuming that $A_n<n!$ for some positive n, we can prove that $A_{n+1}<(n+1)!$ as follows:

$$A_{n+1}=n\cdot(A_n+1)<n\cdot(n!+1)=n\cdot n!+n\le n\cdot n!+n!=(n+1)n!=(n+1)!.$$