Introduction to Mathematical Logic, Handout 2
Propositional Formulas: Semantics

The symbols f and t are called *truth values*. An *interpretation* of a propositional signature σ is a function from σ into $\{\mathsf{f},\mathsf{t}\}$. For instance, an interpretation I of the signature $\{p,q,r\}$ can be defined by the formulas

$$I(p) = \mathsf{f},\ I(q) = \mathsf{f},\ I(r) = \mathsf{t}. \tag{1}$$

The semantics of propositional formulas introduced below defines which truth value is assigned to a formula F by an interpretation I. As a preliminary step, we need to associate functions with all unary and binary connectives: a function from $\{\mathsf{f},\mathsf{t}\}$ into $\{\mathsf{f},\mathsf{t}\}$ with the unary connective $\neg$, and a function from $\{\mathsf{f},\mathsf{t}\} \times \{\mathsf{f},\mathsf{t}\}$ into $\{\mathsf{f},\mathsf{t}\}$ with each of the binary connectives. These functions are denoted by the same symbols as the corresponding connectives, and defined by the following tables:

x	$\neg(x)$
f	t
t	f

x	y	$\wedge(x,y)$	$\vee(x,y)$	$\rightarrow(x,y)$
f	f	f	f	t
f	t	f	t	t
t	f	f	t	f
t	t	t	t	t

For any formula F and any interpretation I, the truth value F^I that is *assigned* to F by I is defined recursively, as follows:

- for any atom F, $F^I = I(F)$,
- $(\neg F)^I = \neg(F^I)$,
- $(F \odot G)^I = \odot(F^I, G^I)$ for every binary connective $\odot$.

If $F^I = \mathsf{t}$ then we say that the interpretation I *satisfies* F and write $I \models F$.

Problem 2.1 (a) Find a formula F of the signature $\{p,q,r\}$ such that (1) is the only interpretation satisfying F. (b) Prove that for any formulas $F_1,\dots,F_n$ ($n \geq 1$) and any interpretation I,

$$(F_1 \wedge \cdots \wedge F_n)^I = \mathsf{t} \text{ iff } F_1^I = \cdots = F_n^I = \mathsf{t},$$
$$(F_1 \vee \cdots \vee F_n)^I = \mathsf{f} \text{ iff } F_1^I = \cdots = F_n^I = \mathsf{f}.$$

In the following two problems, we assume that the underlying signature is finite: $\sigma = \{p_1, \ldots, p_n\}$.

Problem 2.2 For any interpretation I, there exists a formula F such that I is the only interpretation satisfying F.

Problem 2.3 For any function α from interpretations to truth values, there exists a formula F such that, for all interpretations I, $F^I = \alpha(I)$.

A propositional formula F is a *tautology* if every interpretation satisfies F. A formula F is *equivalent* to a formula G (symbolically, $F \sim G$) if, for every interpretation I, $F^I = G^I$.

Problem 2.4 (a) We know that conjunction and disjunction are associative:

$$(F \wedge G) \wedge H \sim F \wedge (G \wedge H),$$
$$(F \vee G) \vee H \sim F \vee (G \vee H).$$

Determine whether equivalence has a similar property:

$$(F \leftrightarrow G) \leftrightarrow H \sim F \leftrightarrow (G \leftrightarrow H).$$

(b) We know that conjunction distributes over disjunction and that disjunction distributes over conjunction:

$$F \wedge (G \vee H) \sim (F \wedge G) \vee (F \wedge H),$$
$$F \vee (G \wedge H) \sim (F \vee G) \wedge (F \vee H).$$

Do these connectives distribute over equivalence? (c) We know that implication distributes over conjunction:

$$F \rightarrow (G \wedge H) \sim (F \rightarrow G) \wedge (F \rightarrow H).$$

Find a similar transformation for $(F \vee G) \rightarrow H$.

Problem 2.5 (a) De Morgan's laws

$$\neg(F \wedge G) \sim \neg F \vee \neg G,$$
$$\neg(F \vee G) \sim \neg F \wedge \neg G$$

show how to transform a formula of the form $\neg(F \odot G)$ when $\odot$ is conjunction or disjunction. Find similar transformations for the cases when $\odot$ is implication or equivalence. (b) To simplify a formula means to find an equivalent formula that is shorter. Simplify the formulas

$$F \vee (F \wedge G),\; F \wedge (F \vee G),\; F \vee (\neg F \wedge G).$$

A set Γ of formulas is *satisfiable* if there exists an interpretation that satisfies all formulas in Γ, and *unsatisfiable* otherwise.

Problem 2.6 For any set Γ of formulas, if every two-element subset of Γ is satisfiable then Γ is satisfiable. True or false?

A *literal* is an atom or the negation of an atom.

Problem 2.7 Let Γ be a set of literals. Show that Γ is satisfiable iff there is no atom A for which both A and $\neg A$ belong to Γ.

A set Γ of formulas *entails* a formula F (symbolically, $\Gamma \models F$), if every interpretation that satisfies all formulas in Γ satisfies F also.

Problem 2.8 For any formulas $F_1, \ldots, F_n, G$, the following conditions are equivalent:

- $F_1, \ldots, F_n \models G$,
- $(F_1 \wedge \cdots \wedge F_n) \to G$ is a tautology,
- the set $\{F_1, \ldots, F_n, \neg G\}$ is unsatisfiable.