Propositional Formulas: Semantics

The symbols \( f \) and \( t \) are called truth values. An interpretation of a propositional signature \( \sigma \) is a function from \( \sigma \) into \( \{ f, t \} \). For instance, an interpretation \( I \) of the signature \( \{ p, q, r \} \) can be defined by the formulas

\[
I(p) = f, \quad I(q) = f, \quad I(r) = t.
\]  

The semantics of propositional formulas introduced below defines which truth value is assigned to a formula \( F \) by an interpretation \( I \). As a preliminary step, we need to associate functions with all unary and binary connectives: a function from \( \{ f, t \} \) into \( \{ f, t \} \) with the unary connective \( \neg \), and a function from \( \{ f, t \} \times \{ f, t \} \) into \( \{ f, t \} \) with each of the binary connectives. These functions are denoted by the same symbols as the corresponding connectives, and defined by the following tables:

\[
\begin{array}{c|c}
\neg(x) & x \\
\hline
f & t \\
\hline
t & f \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
\land (x, y) & \lor (x, y) & \rightarrow (x, y) & x & y \\
\hline
f & f & f & f & t \\
f & t & f & t & t \\
t & f & f & t & f \\
t & t & t & t & t \\
\end{array}
\]

For any formula \( F \) and any interpretation \( I \), the truth value \( F^I \) that is assigned to \( F \) by \( I \) is defined recursively, as follows:

- for any atom \( F \), \( F^I = I(F) \),
- \( (\neg F)^I = \neg (F^I) \),
- \( (F \circ G)^I = \circ (F^I, G^I) \) for every binary connective \( \circ \).

If \( F^I = t \) then we say that the interpretation \( I \) satisfies \( F \) and write \( I \models F \).

**Problem 2.1** (a) Find a formula \( F \) of the signature \( \{ p, q, r \} \) such that (1) is the only interpretation satisfying \( F \). (b) Prove that for any formulas \( F_1, \ldots, F_n \) \( (n \geq 1) \) and any interpretation \( I \),

\[
(F_1 \land \cdots \land F_n)^I = t \text{ iff } F_1^I = \cdots = F_n^I = t,
\]

\[
(F_1 \lor \cdots \lor F_n)^I = f \text{ iff } F_1^I = \cdots = F_n^I = f.
\]
In the following two problems, we assume that the underlying signature is finite: \( \sigma = \{ p_1, \ldots, p_n \} \).

**Problem 2.2** For any interpretation \( I \), there exists a formula \( F \) such that \( I \) is the only interpretation satisfying \( F \).

**Problem 2.3** For any function \( \alpha \) from interpretations to truth values, there exists a formula \( F \) such that, for all interpretations \( I \), \( F^I = \alpha(I) \).

A propositional formula \( F \) is a **tautology** if every interpretation satisfies \( F \). A formula \( F \) is **equivalent** to a formula \( G \) (symbolically, \( F \sim G \)) if, for every interpretation \( I \), \( F^I = G^I \).

**Problem 2.4** (a) We know that conjunction and disjunction are associative:

\[
(F \land G) \land H \sim F \land (G \land H),
(F \lor G) \lor H \sim F \lor (G \lor H).
\]

Determine whether equivalence has a similar property:

\[
(F \leftrightarrow G) \leftrightarrow H \sim F \leftrightarrow (G \leftrightarrow H).
\]

(b) We know that conjunction distributes over disjunction and that disjunction distributes over conjunction:

\[
F \land (G \lor H) \sim (F \land G) \lor (F \land H),
F \lor (G \land H) \sim (F \lor G) \land (F \lor H).
\]

Do these connectives distribute over equivalence? (c) We know that implication distributes over conjunction:

\[
F \rightarrow (G \land H) \sim (F \rightarrow G) \land (F \rightarrow H).
\]

Find a similar transformation for \( (F \lor G) \rightarrow H \).

**Problem 2.5** (a) De Morgan’s laws

\[
\neg(F \land G) \sim \neg F \lor \neg G,
\neg(F \lor G) \sim \neg F \land \neg G
\]

show how to transform a formula of the form \( \neg(F \odot G) \) when \( \odot \) is conjunction or disjunction. Find similar transformations for the cases when \( \odot \) is implication or equivalence. (b) To simplify a formula means to find an equivalent formula that is shorter. Simplify the formulas

\[
F \lor (F \land G), \ F \land (F \lor G), \ F \lor (\neg F \land G).
\]
A set $\Gamma$ of formulas is \textit{satisfiable} if there exists an interpretation that satisfies all formulas in $\Gamma$, and \textit{unsatisfiable} otherwise.

**Problem 2.6** For any set $\Gamma$ of formulas, if every two-element subset of $\Gamma$ is satisfiable then $\Gamma$ is satisfiable. True or false?

A \textit{literal} is an atom or the negation of an atom.

**Problem 2.7** Let $\Gamma$ be a set of literals. Show that $\Gamma$ is satisfiable iff there is no atom $A$ for which both $A$ and $\neg A$ belong to $\Gamma$.

A set $\Gamma$ of formulas \textit{entails} a formula $F$ (symbolically, $\Gamma \models F$), if every interpretation that satisfies all formulas in $\Gamma$ satisfies $F$ also.

**Problem 2.8** For any formulas $F_1, \ldots, F_n, G$, the following conditions are equivalent:

- $F_1, \ldots, F_n \models G$,
- $(F_1 \land \cdots \land F_n) \rightarrow G$ is a tautology,
- the set $\{F_1, \ldots, F_n, \neg G\}$ is unsatisfiable.