Introduction to Mathematical Logic, Handout 3

Adequate Sets of Connectives, Normal Forms, and Clausification

Problem 3.1 For any formula, there exists an equivalent formula that contains no connectives other than (i) conjunction and negation; (ii) disjunction and negation; (iii) implication and negation.

Problem 3.2 Any propositional formula equivalent to $\neg p$ contains negation.

A propositional formula is said to be in *negation normal form* if

- it contains no connectives other that conjunction, disjunction, and negation, and
- every negation in it is part of a literal.

Problem 3.3 Any formula is equivalent to a formula in negation normal form.

A *simple conjunction* is a formula of the form $L_1 \wedge \cdots \wedge L_n$ $(n \geq 1)$, where $L_1, \ldots, L_n$ are literals. A formula is in *disjunctive normal form (DNF)* if it has the form $C_1 \vee \cdots \vee C_m$ $(m \geq 1)$, where $C_1, \ldots, C_m$ are simple conjunctions.

Problem 3.4 Any formula is equivalent to a formula in disjunctive normal form.

A *simple disjunction* is a formula of the form $L_1 \vee \cdots \vee L_n$ $(n \geq 1)$, where $L_1, \ldots, L_n$ are literals. (Simple disjunctions are also called *clauses*.) A formula is in *conjunctive normal form (CNF)* if it has the form $D_1 \wedge \cdots \wedge D_m$ $(m \geq 1)$, where $D_1, \ldots, D_m$ are simple disjunctions.

Problem 3.5 Let F be a formula in disjunctive normal form. Show that $\neg F$ is equivalent to a formula in conjunctive normal form.

Problem 3.6 Any formula is equivalent to a formula in conjunctive normal form.

To *clausify* a formula F of a signature σ means to find a formula F' of some signature σ' containing σ such that

- F' is in conjunctive normal form,
- any interpretation I of σ satisfying F can be extended to an interpretation I' of σ' that satisfies F',
- for any interpretation I' of σ' satisfying F', the restriction of I' to σ satisfies F.

Here is an algorithm for clausifying a propositional formula:

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CLAUSIFY(F);
begin
  Γ ← ∅;
  while F is not CNF do
    A ← a new atom;
    G ← a minimal non-literal subformula of F;
    F ← the result of replacing G in F by A;
    Δ ← the set of clauses of the CNF of A ↔ G;
    Γ ← Γ ∪ Δ;
  return the conjunction of F with clauses Γ;
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Problem 3.7 (a) Apply the algorithm CLAUSIFY to the formula

$$p \vee \neg(q \to r). \qquad (1)$$

(b) Determine whether (1), viewed as a formula of the extended signature, is equivalent to the result of its clausification.