Introduction to Mathematical Logic, Handout 4
Natural Deduction

A *sequent* is an expression of the form

$$\Gamma \Rightarrow F \tag{1}$$

("F under assumptions Γ") or

$$\Gamma \Rightarrow \tag{2}$$

("assumptions Γ are contradictory"), where Γ is a finite set of formulas. If Γ is written as $\{G_1, \ldots, G_n\}$, we will drop the braces and write (1) as

$$G_1, \ldots, G_n \Rightarrow F \tag{3}$$

and (2) as

$$G_1, \ldots, G_n \Rightarrow . \tag{4}$$

Intuitively, a sequent (3) has the same meaning as the formula

$$(G_1 \wedge \cdots \wedge G_n) \rightarrow F \tag{5}$$

(and as the formula F if $n = 0$); (4) has the same meaning as the formula

$$\neg(G_1 \wedge \cdots \wedge G_n). \tag{6}$$

We define below which sequents are considered *axioms* and provide a list of *inference rules*. A *proof* is a list of sequents $S_1, \ldots, S_n$ such that each S_i is either an axiom or can be derived from some of the sequents $S_1, \ldots, S_{i-1}$ by one of the inference rules.

Axioms are sequents of the forms

$$F \Rightarrow F$$

and

$$\Rightarrow F \vee \neg F.$$

Inference Rules. In the list below, Γ, Δ, Δ_1, Δ_2 are finite sets of formulas, and Σ is a formula or the empty string. Most inference rules are classified into *introduction rules* (the left column) and *elimination rules* (the right column); two exceptions are the *contradiction rule* (C) and the *weakening rule* (W).

$(\wedge I)\ \dfrac{\Gamma \Rightarrow F \quad \Delta \Rightarrow G}{\Gamma, \Delta \Rightarrow F \wedge G}$ $\qquad$ $(\wedge E)\ \dfrac{\Gamma \Rightarrow F \wedge G}{\Gamma \Rightarrow F} \quad \dfrac{\Gamma \Rightarrow F \wedge G}{\Gamma \Rightarrow G}$

$(\vee I)\ \dfrac{\Gamma \Rightarrow F}{\Gamma \Rightarrow F \vee G} \quad \dfrac{\Gamma \Rightarrow G}{\Gamma \Rightarrow F \vee G}$ $\qquad$ $(\vee E)\ \dfrac{\Gamma \Rightarrow F \vee G \quad \Delta_1, F \Rightarrow \Sigma \quad \Delta_2, G \Rightarrow \Sigma}{\Gamma, \Delta_1, \Delta_2 \Rightarrow \Sigma}$

$(\to I)\ \dfrac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \to G}$ $\qquad$ $(\to E)\ \dfrac{\Gamma \Rightarrow F \quad \Delta \Rightarrow F \to G}{\Gamma, \Delta \Rightarrow G}$

$(\neg I)\ \dfrac{\Gamma, F \Rightarrow}{\Gamma \Rightarrow \neg F}$ $\qquad$ $(\neg E)\ \dfrac{\Gamma \Rightarrow F \quad \Delta \Rightarrow \neg F}{\Gamma, \Delta \Rightarrow}$

$$(C)\ \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow F}$$

$$(W)\ \frac{\Gamma \Rightarrow \Sigma}{\Gamma, \Delta \Rightarrow \Sigma}$$

To prove a sequent S means to find a proof with the last sequent S. To prove a formula F means to prove the sequent $\Rightarrow F$. For instance, here is a proof of the formula $(p \wedge q) \to (p \vee q)$:

$$\begin{array}{l} p \wedge q \Rightarrow p \wedge q \\ p \wedge q \Rightarrow p \\ p \wedge q \Rightarrow p \vee q \\ \Rightarrow (p \wedge q) \to (p \vee q) \end{array}$$

In each of the following problems, find a proof of the given formula.

Problem 4.1 $(p \wedge q \wedge r) \to (p \wedge r)$.

Problem 4.2 $((p \wedge q) \to r) \to (p \to (q \to r))$.

Problem 4.3 $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$.

Problem 4.4 $(p \to q) \to (\neg q \to \neg p)$.

Problem 4.5 $(p \wedge \neg p) \to q$.

Problem 4.6 $((p \wedge q) \vee r) \to (p \vee r)$.

Problem 4.7 $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$.

Problem 4.8 $p \to (q \to p)$.

Problem 4.9 $\neg\neg p \leftrightarrow p$.

Problem 4.10 $(p \to q) \vee (q \to p)$.

Problem 4.11 $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$.

Problem 4.12 $(p \vee q) \leftrightarrow (\neg p \to q)$.