Introduction to Mathematical Logic, Handout 8
Introduction and Elimination Rules for Quantifiers

We say that a term t is *substitutable* for a variable v in a formula F if

- t is a constant, or
- t is a variable w, and no part of F of the form KwG contains an occurrence of v which is free in F.

Here are the additional inference rules of predicate logic:

$(\forall I)\ \dfrac{\Gamma \Rightarrow F}{\Gamma \Rightarrow \forall v F}$

where v is not a free variable of any formula in Γ

$(\forall E)\ \dfrac{\Gamma \Rightarrow \forall v F}{\Gamma \Rightarrow F_t^v}$

where t is substitutable for v in F

$(\exists I)\ \dfrac{\Gamma \Rightarrow F_t^v}{\Gamma \Rightarrow \exists v F}$

where t is substitutable for v in F

$(\exists E)\ \dfrac{\Gamma \Rightarrow \exists v F \quad \Delta, F \Rightarrow \Sigma}{\Gamma, \Delta \Rightarrow \Sigma}$

where v is not a free variable of any formula in Δ, Σ

Prove the given formulas in the natural deduction system.

Problem 8.1 $(P(a) \wedge \forall x(P(x) \to Q(x))) \to Q(a)$.

Problem 8.2 $P(a) \to \neg\forall x \neg P(x)$.

Problem 8.3 $\forall xy P(x,y) \to \forall x P(x,x)$.

Problem 8.4 $\forall x P(x) \leftrightarrow \forall y P(y)$.

Problem 8.5 $\forall x P(x) \wedge \forall x Q(x) \leftrightarrow \forall x(P(x) \wedge Q(x))$.

Problem 8.6 $(P(a) \vee P(b)) \to \exists x P(x)$.

Problem 8.7 $\exists x(P(x) \vee Q(x)) \leftrightarrow \exists x P(x) \vee \exists x Q(x)$.

Problem 8.8 $(\exists x P(x) \wedge \forall x(P(x) \to Q(x))) \to \exists x Q(x)$.

Problem 8.9 $\exists x P(x) \leftrightarrow \exists y P(y)$.

Problem 8.10 $\neg\exists x P(x) \leftrightarrow \forall x \neg P(x)$.

Problem 8.11 $\exists x(P(x) \wedge Q(a)) \leftrightarrow \exists x P(x) \wedge Q(a)$.

Problem 8.12 $\forall x P(x) \vee Q(a) \leftrightarrow \forall x(P(x) \vee Q(a))$.