Introduction to Mathematical Logic, Handout 9
First-Order Logic: Function Symbols and Equality

The concept of a predicate signature (Handout 6) can be generalized as follows. A *signature* is a set of symbols of three kinds—*object constants*, *function constants*, and *predicate constants*—with a positive integer, called the *arity*, assigned to every function constant and to every predicate constant. *Terms* of such a signature are defined recursively:

- every object constant is a term,
- every object variable is a term,
- if $t_1, \dots, t_n$ are terms and f is a function constant of arity n then $f(t_1, \dots, t_n)$ is a term.

The class of atomic formulas includes, in addition to expressions of the form

$$P(t_1, \dots, t_n)$$

as in Handout 6, expressions of a second kind: *equalities*

$$(t_1 = t_2)$$

where t_1, t_2 are terms. Otherwise, the definition of a formula remains the same. The expression $t_1 \neq t_2$ is shorthand for $\neg(t_1 = t_2)$.

As an example, consider the *signature of first-order arithmetic*

$$\{a, s, f, g\}, \tag{1}$$

where a is an object constant (intended to represent 0), s is a unary function constant (for the successor function), and f, g are binary function constants (for addition and multiplication). Since this signature includes no predicate constants, its only atomic formulas are equalities.

Problem 9.1 Represent the following English sentences by first-order formulas:

- There exists at most one x such that $P(x)$.
- There exists exactly one x such that $P(x)$.
- There exist at least two x such that $P(x)$.

- There exist at most two x such that $P(x)$.
- There exist exactly two x such that $P(x)$.

For a signature containing function constants, an *interpretation* I consists of

- a non-empty set $|I|$, called the *universe* of I,
- for every object constant c of σ, an element c^I of $|I|$,
- for every function constant f of σ, a function f^I from $|I|^n$ to $|I|$, where n is the arity of f,
- for every predicate constant P of σ, a function P^I from $|I|^n$ to $\{\mathsf{f}, \mathsf{t}\}$, where n is the arity of P.

For example, the intended interpretation I of (1) is defined as follows:

$$\begin{array}{l} |I| = \mathbf{N}, \\ a^I = 0, \\ s^I(n) = n + 1, \\ f^I(m, n) = m + n, \\ g^I(m, n) = m \cdot n. \end{array} \tag{2}$$

In this more general setting, a term t without variables is not necessarily an object constant; such a term may contain function constants. The notation t^I is extended to these more complex terms by the recursive equation

$$f(t_1, \ldots, t_n)^I = f^I(t_1^I, \ldots, t_n^I).$$

The recursive definition of F^I is extended by a clause for equalities:

$$(t_1 = t_2)^I = \mathsf{t} \text{ iff } t_1^I = t_2^I.$$

Problem 9.2 For each of the following sentences determine whether it is satisfiable:

- $a = b$,
- $\forall xy(x = y)$,
- $\forall xy(x \neq y)$.

In the presence of function symbols, the definition of a substitutable term is stated as follows: A term t is *substitutable* for a variable v in a formula F if, for each variable w occurring in t, no part of F of the form KwG contains an occurrence of v which is free in F.

In the presence of equality, the natural deduction system is extended by the axioms

$$\Rightarrow t = t$$

for any term t, and by two inference rules:

$$(R)\ \frac{\Gamma \Rightarrow t_1 = t_2 \quad \Delta \Rightarrow F^v_{t_1}}{\Gamma, \Delta \Rightarrow F^v_{t_2}} \quad \frac{\Gamma \Rightarrow t_1 = t_2 \quad \Delta \Rightarrow F^v_{t_2}}{\Gamma, \Delta \Rightarrow F^v_{t_1}}$$

where t_1 and t_2 are substitutable for v in F.

Prove the given formulas in the natural deduction system.

Problem 9.3 $x = y \to f(x, y) = f(y, x)$.

Problem 9.4 $\forall x \exists y (y = f(x))$.

Problem 9.5 $\exists y (x = y \wedge y = z) \to x = z$.

Problem 9.6 $(\exists x P(x) \wedge \exists x \neg P(x)) \to \exists xy (x \neq y)$.

Problem 9.7 $\forall x (x = a \to P(x)) \leftrightarrow P(a)$.

Problem 9.8 $\exists x (x = a \wedge P(x)) \leftrightarrow P(a)$.