Reducing Graph Coloring to SAT

A \( k \)-coloring of a graph is a labelling of its vertices with at most \( k \) colors such that no two vertices sharing the same edge have the same color. The problem of generating a \( k \)-coloring of a graph \((V, E)\) can be reduced to SAT as follows. For every \( v \in V \) and every \( i \in \{1, \ldots, k\} \), introduce an atom \( p_{vi} \). Intuitively, this atom expresses that vertex \( v \) is assigned color \( i \). Consider the following propositional formulas:

\[
\bigvee_{1 \leq i \leq k} p_{vi} \quad (v \in V),
\]

\[
\neg(p_{vi} \land p_{vj}) \quad (v \in V, 1 \leq i < j \leq k),
\]

\[
\neg(p_{vi} \land p_{wi}) \quad ({\{v, w}\} \in E, 1 \leq i \leq k).
\]

The interpretations satisfying these formulas are in a 1–1 correspondence with \( k \)-colorings of \((V, E)\).

**Problem 3.** (a) Write out formulas (1) for the graph

\[
A \quad \longrightarrow \quad B \quad \longrightarrow \quad C
\]

and \( k = 2 \). (Suggestion: use the abbreviation \( p_{A1} \) for \( A1 \), and similarly for the other atoms.) (b) We would like to find a \( k \)-coloring of a graph \((V, E)\) such that color 1 is assigned to at most one vertex. Modify formulas (1) accordingly.

**Problem 4.** Use DPLL to find (a) a 2-coloring of the graph from Problem 3; (b) a 2-coloring of that graph such that color 1 is assigned to at most one vertex.