Program Completion

The definitions below are based on [Clark, 1978] and [Lloyd and Topor, 1984]. We begin with a finite signature \( \sigma \) (in the sense of first-order logic) that has no function constants of arity \( > 0 \).

A rule is a first-order formula of \( \sigma \) that has the form

\[
F \rightarrow P(t),
\]

where \( F \) is a formula, \( P \) a predicate constant, and \( t \) a tuple of terms. We will usually write (1) as

\[
P(t) \leftarrow F
\]

and call \( P(t) \) the head of the rule, and \( F \) its body. We will identify an atomic formula \( P(t) \) with the rule \( P(t) \leftarrow \top \).

A logic program is a finite set of rules.

The completion formula for an \( n \)-ary predicate constant \( P \) relative to a logic program \( \Pi \) is the sentence obtained as follows:

(i) choose \( n \) variables \( x_1, \ldots, x_n \) that are pairwise distinct and do not occur in \( \Pi \);

(ii) for each rule

\[
P(t_1, \ldots, t_n) \leftarrow F
\]

of \( \Pi \) containing \( P \) in the head, form the rule

\[
P(x_1, \ldots, x_n) \leftarrow F \land x_1 = t_1 \land \cdots \land x_n = t_n;
\]

(iii) for each of the rules

\[
P(x) \leftarrow F
\]

obtained on the previous step, make the list \( y \) of all variables that occur in its body \( F \) but do not occur in its head \( P(x) \), and replace the body \( F \) of the rule with \( \exists y F \);

(iv) take all rules

\[
P(x) \leftarrow F_i \quad (i = 1, \ldots, k)
\]

obtained on the previous step, and form the sentence

\[
\forall x (P(x) \leftarrow F_1 \lor \cdots \lor F_k).
\]
The completion of a logic program $\Pi$ is the set consisting of the completion sentences for all predicate constants of $\sigma$ relative to $\Pi$ and the formulas
\[ c_i \neq c_j \quad (1 \leq i < j \leq m), \]
where $c_1, \ldots, c_m$ are the object constants of $\sigma$. (These formulas are said to express the unique name assumption.)

**Problem 13.** (a) Form the completion of the program consisting of the rules
\[ P(a), \quad Q(b). \quad (2) \]
(b) Show that the formulas $\neg P(b)$, $\neg Q(a)$ are not entailed by the rules of this program, but are entailed by its completion.

**Problem 14.** (a) Form the completion of the program consisting of the rules
\[
\begin{align*}
R(x) &\leftarrow P(x), \\
R(x) &\leftarrow Q(x)
\end{align*}
\quad (3)
\]
and simplify it as much as possible. (b) Do the same for the program consisting of rules (2) and (3).

**Problem 15.** For each of the given logic programs, find all models of its completion:
(a) $P \leftarrow \neg Q$;
(b) $P \leftarrow \neg Q$, $Q \leftarrow \neg R$;
(c) $P \leftarrow \neg Q$, $Q \leftarrow \neg P$.

**References**
