Stable Models of Positive Programs, Part 1

From now on, we will assume that bodies of rules contain neither → nor ↔. In other words, bodies of rules will be first-order formulas built from atomic formulas and the 0-place connectives \( \top, \bot \) using \( \land, \lor, \neg, \forall, \exists \).

A logic program is positive if every negation in it is applied to an equality. For instance, the rule

\[
\text{Brother}(x, y) \leftarrow \text{Parent}(z, x) \land \text{Parent}(z, y) \land \text{Male}(x) \land x \neq y
\]

is positive, and the rule

\[
\text{Female}(x) \leftarrow \neg \text{Male}(x) \land \text{Person}(x)
\]

is not.

We will first define the concept of a stable model for positive programs containing a single predicate constant, such as

\[
P(a), \ P(b), \tag{1}
\]

or

\[
P(a, b),
\]

\[
P(x, y) \leftarrow P(y, x), \tag{2}
\]

or

\[
P(a, b),
P(b, c),
P(x, y) \leftarrow P(x, z) \land P(z, y). \tag{3}
\]

The rules of program (1) express that the set \( P \) contains \( a \) and \( b \), possibly along with other elements. The corresponding completion formula

\[
\forall x(P(x) \leftrightarrow x = a \lor x = b) \tag{4}
\]

is stronger: it expresses that \( P \) is the set \( \{a, b\} \). This set is the smallest among the sets containing \( a \) and \( b \); in other words, it is the intersection of all sets containing \( a \) and \( b \). The condition

\[
x \text{ belongs to the intersection of all sets containing } a \text{ and } b
\]

\[
1
\]
can be expressed by the second-order formula
\[ \forall p (p(a) \land p(b) \rightarrow p(x)). \quad (5) \]
Consequently, (4) is equivalent to the formula
\[ \forall x (P(x) \leftrightarrow \forall p (p(a) \land p(b) \rightarrow p(x))). \quad (6) \]

This example motivates the following definition. Let \( \Pi \) be a positive program containing no predicate constants other than \( P \). By \( \Pi^\circ(p) \) we denote the formula obtained by forming the conjunction of the universal closures of the rules of \( \Pi \) and then replacing each occurrence of \( P \) in this conjunction by the predicate variable \( p \). The sentence
\[ \forall x(P(x) \leftrightarrow \forall p(\Pi^\circ(p) \rightarrow p(x))), \]
where \( x \) is a tuple of distinct object variables, will be called the stability formula for \( P \) relative to \( \Pi \). This formula expresses that \( P \) is the smallest of the sets \( p \) satisfying \( \Pi^\circ(p) \).

**Examples.** If \( \Pi \) is (1) then \( \Pi^\circ(p) \) is \( p(a) \land p(b) \), so that the stability formula in this case is (6). In case of program (2), the stability formula is
\[ \forall uv(P(u, v) \leftrightarrow \forall p(p(a, b) \land \forall xy (p(y, x) \rightarrow p(x, y)) \rightarrow p(u, v))). \quad (7) \]
This formula expresses that \( P \) is the smallest of the symmetric binary relations \( p \) satisfying \( p(a, b) \). Consequently (7) is equivalent to
\[ \forall uv(P(u, v) \leftrightarrow (u = a \land v = b) \lor (u = b \land v = a)). \]
In case of program (3), the stability formula is
\[ \forall uv(P(u, v) \leftrightarrow \forall p(p(a, b) \land p(b, c) \land \forall xyz (p(x, z) \land p(z, y) \rightarrow p(x, y)) \rightarrow p(u, v))). \quad (8) \]
This formula expresses that \( P \) is the smallest of the transitive binary relations \( p \) satisfying \( p(a, b) \) and \( p(b, c) \), or, in other words, that \( P \) is the transitive closure of the relation
\[ \{ (a, b), (b, c) \}. \]
Consequently (8) is equivalent to
\[ \forall uv(P(u, v) \leftrightarrow (u = a \land v = b) \lor (u = b \land v = c) \lor (u = a \land v = c)). \]

For any positive program containing no predicate constants other than \( P \), the stability formula for \( P \) entails the completion formula for \( P \). For some programs, for
instance (1), the two formulas are equivalent to each other. But there are also cases when the stability formula is stronger.

**Problem 21.** Determine whether stability is equivalent to completion in the case of program (2).

**Problem 22.** Determine whether stability is equivalent to completion in the case of program (3).

Recall that an *Herbrand interpretation* (of a signature without function constants of nonzero arity) is an interpretation $I$ such that

- the universe of $I$ is the set of all object constants, and
- every object constant is interpreted by $I$ as itself.

An Herbrand interpretation can be identified with the set of ground atoms that are true in it.

For any positive program containing no predicate constants other than $P$, its *stable model* is the Herbrand interpretation satisfying the stability formula for $P$. Every positive program has a unique stable model (even a program containing several predicate constants, as will be discussed later.)

**Problem 23c.** Consider the program

\[
P(a, b),
\]

\[
P(b, c),
\]

\[
P(d, d),
\]

\[
P(x, y) \leftarrow P(y, x),
\]

\[
P(x, y) \leftarrow P(x, z) \land P(z, y).
\]

(a) Make the list of ground atoms that are true in the stable model of this program.

(b) Use CLINGO to check that your answer is correct.