Monotonicity and quantification

Predicate transformer f is monotonic means

$$[X \Rrightarrow Y] \Rightarrow [f X \Rrightarrow f Y] \quad .$$

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Let $X i$ and $Y i$ stand for predicates on some space S for any i from a domain D. Then $X i$ and $Y i$ define two predicates $\tilde{X}$ and $\tilde{Y}$ respectively on the Cartesian product $S \times D$, according to

$$[\tilde{X}(I/i) \equiv X\, I] \quad \text{for all } I \text{ in } D \ .$$

(Here, the square brackets denote universal quantification over S; $\tilde{X}(I/i)$ denotes $\tilde{X}$ with I substituted for i, the formal definition of which is given by $[\tilde{X}(I/i) \equiv (\underline{A}\, i :: i = I : \tilde{X})]$.)

In this notation $[\tilde{X} \Rrightarrow \tilde{Y}]$ is defined by

$$[\tilde{X} \Rrightarrow \tilde{Y}] = (\underline{A}\, i :: i \ \underline{in}\ D : [X i \Rrightarrow Y i]) \quad .$$

(According to our habits and conventions, the square brackets in the left-hand side are over $S \times D$, those in the right-hand side over S.)

We draw attention to the fact that universal quantification and existential quantification are monotonic, i.e.

$$[\tilde{X} \Rrightarrow \tilde{Y}] \Rightarrow [(\underline{A}\, i :: X i) \Rrightarrow (\underline{A}\, i :: Y i)]$$

and

$$[\tilde{X} \Rrightarrow \tilde{Y}] \Rightarrow [(\underline{E}\, i :: X i) \Rrightarrow (\underline{E}\, i :: Y i)] \quad .$$

(Note that quantification over i represents a function from predicates over $S \times D$ to predicates over S.)

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Let Xi stand for a predicate on some space for any i from some domain. We then have for any I from that domain

$$[(\underline{A}\, i :: X i) \Rrightarrow X I] \quad ;$$

hence we have for monotonic f for any I

$$[f(\underline{A}\, i :: X i) \Rrightarrow f(X I)] \quad ,$$

and since this holds for any I from the domain, we conclude for monotonic f

$$[f(\underline{A}\, i :: X i) \Rrightarrow (\underline{A}\, i :: f(X i))] \qquad (0).$$

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Let Yj stand for a predicate on some space for any j from some domain. We then have for any J from that domain

$$[Y J \Rrightarrow (\underline{E}\, j :: Y j)] \quad ;$$

hence we have for monotonic h for any J

$$[h(Y J) \Rrightarrow h(\underline{E}\, j :: Y j)] \quad ,$$

and since this holds for any J from the domain, we conclude for monotonic h

$$[(\underline{E}\, j :: h(Y j)) \Rrightarrow h(\underline{E}\, j :: Y j)] \qquad . \quad (1)$$

For some obscure reason, formula (1) is less well known than formula (0). Both formulae are of importance, since they allow us to deal with predicate transformers that are monotonic but of whose junctivity properties we know nothing.

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Finally, we combine these two results. Let, for any i from D_i and any j from D_j, $Z\,i\,j$ be a predicate on some space S.

Let then, for any i from D_i, the predicate $X\,i$ be a predicate on $S \times D_j$ defined by

$$[(X\,i)(J/j) \equiv Z\,i\,J] \quad \text{for any } J \text{ from } D_j.$$

Applying (0) with for f existential quantification over j — which is monotonic — yields

$$[(\underline{E}j :: (\underline{A}i :: Z\,i\,j)) \Rrightarrow (\underline{A}i :: (\underline{E}j :: Z\,i\,j))] \qquad (2)$$

Alternatively, we could have defined, for any j from D_j, $Y\,j$ to be a predicate on $S \times D_i$, satisfying $[(Y\,j)(I/i) \equiv Z\,I\,j]$ for any I from D_i.

Applying (1) with for h universal quantification over i — which is monotonic — also yields (2). Had we chosen for h existential quantification over i, we would have gotten from (1)

$$[(\underline{E}j :: (\underline{E}i :: Z\,i\,j)) \Rrightarrow (\underline{E}i :: (\underline{E}j :: Z\,i\,j))] \quad ;$$

with the inverse implication holding for reasons of symmetry, we get

$$[(\underline{E}j :: (\underline{E}i :: Z\,i\,j)) \equiv (\underline{E}i :: (\underline{E}j :: Z\,i\,j))] \quad ,$$

i.e. existential quantifications commute. Similarly, one can derive from (o) that universal quantifications commute.

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<u>Notational Remark</u>. In the introduction I had to distinguish between the predicates $X\,i$ on S and the predicate $\tilde{X}$ on $S \times D$, with D the domain of i. I wrote the somewhat pompous

$$\text{“}\ [\tilde{X}(I/i) \equiv X\,I] \ \text{ for all } I \text{ in } D\text{”} \quad ,$$

not quite daring to write the so much simpler

$$[\tilde{X} \equiv X\,i] \quad .$$

Here the square brackets denote universal quantification over $S \times D$. But this is tricky. We are supposed to "understand" — or: to "remember" — that $\tilde{X}$ contains i as free variable and we are confronted with two conflicting interpretations of $X\,i$: for any i a predicate on S versus a predicate on $S \times D$.

What we call "a predicate on S" is something of the type

$$S \dot{\rightarrow} \text{bool} \quad ,$$

which stands for a boolean-valued expression in in the coordinates of S'; these coordinates are deemed to be named, though the names have been left unmentioned.

When we write that "for any i from D, $X\,i$ stands for a predicate on S", we probably mean that X is a function of the type

$$D \rightarrow (S \dot{\rightarrow} \text{bool}) \quad .$$

Were I to explain $\tilde{X}$, I would probably say that it was something of the type

$$(D_i \times S) \dot{\rightarrow} \text{bool} \quad ,$$

the subscript on D mentioning the name of the free variable i that has entered the boolean-valued expression. Should we make a notational distinction between the two interpretations of $X\,i$ and indicate, for instance, explicitly the coordinate with which the space has been extended, e.g.

$$[\tilde{X} \equiv (\underline{\text{var}}\ i : i\ \underline{\text{in}}\ D : X\,i)] \quad ?$$

(End of Notational Remark.)

Turned out to be a notational question!

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30 January 1984

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