



Proof Reflexivity follows from the idempotence of  $\parallel$ , transitivity follows from the associativity of  $\parallel$ , and antisymmetry follows from the symmetry of  $\parallel$ .  
(End of Proof.)

Theorem 1 The  $\parallel$  is monotonic, i.e.

$$a \leq b \Rightarrow a \parallel c \leq b \parallel c$$

Proof  $a \leq b$   
 = {def. of  $\leq$ }  
 $a \parallel b = b$   
 $\Rightarrow$  {Leibniz?}  
 $a \parallel b \parallel c = b \parallel c$   
 = {properties of  $\parallel$ }  
 $(a \parallel c) \parallel (b \parallel c) = (b \parallel c)$   
 = {def. of  $\leq$ }  
 $a \parallel c \leq b \parallel c$  . (End of Proof.)

We introduce the constant  $0$  as special regular expression: it is the unit element of  $\parallel$ :

Axiom 2 The expression  $0$  is regular and satisfies  $0 \parallel a = a$  or, equivalently,  $0 \leq a$  .

Our second constructor, called "concatenation", indicated by juxtaposition and not pronounced, is introduced by

Axiom 3 The expression  $ab$  is regular. The (invisible) infix operator is associative, i.e.

$$(ab)c = a(bc)$$

Syntactic convention. In view of the associativity of concatenation we allow ourselves the omission of parentheses. Concatenation has a higher binding power than the  $\parallel$ , i.e.  $ab \parallel c = (ab) \parallel c$ . (End of Syntactic convention.)

Axiom 4 Concatenation distributes in both directions over the  $\parallel$ , i.e.

$$(a \parallel b)c = ac \parallel bc$$

$$a(b \parallel c) = ab \parallel ac$$

Theorem 2. Concatenation is monotonic in both its arguments, i.e.

$$a \leq b \Rightarrow ac \leq bc$$

$$b \leq c \Rightarrow ab \leq ac$$

Corollary 0.  $a \leq b \wedge c \leq d \Rightarrow ac \leq bd$

<u>Proof</u>	$a \leq b$	$b \leq c$
	= {def. of $\leq$ }	= {def. of $\leq$ }
	$a \parallel b = b$	$b \parallel c = c$
	$\Rightarrow$ {Leibniz}	$\Rightarrow$ {Leibniz}
	$(a \parallel b)c = bc$	$a(b \parallel c) = ac$
	= {Axiom 4}	= {Axiom 4}
	$ac \parallel bc = bc$	$ab \parallel ac = ac$
	= {def. of $\leq$ }	= {def. of $\leq$ }
	$ac \leq bc$	$ab \leq ac$

(End of Proof of Theorem 2.)

Axiom 5  $0a = 0$  and  $a0 = 0$

For concatenation, a single unit element is introduced; we denote it by  $1$ , which will shortly be recognized as an abbreviation.

Axiom 6 The expression  $1$  is regular and satisfies  $1a = a$  and  $a1 = a$ .

(Expressions  $0$  and  $1$  differ from each other and from all the letters.)

Our last constructor, called "closure", indicated by a postfix  $*$  with highest binding power and pronounced "star", is introduced by

Axiom 7  $(a \parallel b)^* = (a^* b^*)^*$

Axiom 8  $(ab)^* = 1 \parallel a (ba)^* b$

Theorem 3  $0^* = 1$

Proof true

= {Ax. 8 with  $b := 0$ }

$(a0)^* = 1 \parallel a (0a)^* 0$

= {Ax. 5, twice}

$0^* = 1 \parallel 0$

= {Ax. 2}

$0^* = 1$

(End of Proof.)

Theorem 3 justifies our characterization of  $1$  as "an abbreviation".

Theorem 4 The  $*$  is idempotent, i.e.  $a^* = a^{**}$ .

Proof true  
 $= \{ \text{Axiom 7 with } b := 0 \}$   
 $(a \parallel 0)^* = (a^* 0^*)^*$   
 $= \{ \text{Axiom 2 and Theorem 3} \}$   
 $a^* = (a^* 1)^*$   
 $= \{ \text{Axiom 6} \}$   
 $a^* = a^{**}$  . (End of Proof.)

Theorem 4 justifies the name "closure".

Theorem 5  $1 = 1^*$ .

Proof true  
 $= \{ \text{Theorem 4 with } a := 0 \}$   
 $0^* = 0^{**}$   
 $= \{ \text{Theorem 3} \}$   
 $1 = 1^*$  . (End of Proof.)

Theorem 6  $a^* = 1 \parallel a a^*$  and  $b^* = 1 \parallel b^* b$ .

Proof From Axiom 8 by  $b := 1$  and  $a := 1$  respectively, and Axiom 6. (End of Proof)

Theorem 7  $a^* = a^* a^*$ .

Proof true  
 $= \{ \text{Theorem 6 with } a := a^* \}$   
 $a^{**} = 1 \parallel a^* a^{**}$   
 $= \{ \text{Theorem 4} \}$   
 $a^* = 1 \parallel a^* a^*$   
 $= \{ \text{Axiom 6} \}$   
 $a^* = 1 1 \parallel a^* a^*$

{Corollary 0 and  $1 \leq a^*$  from Theorem 6}  
 $a^* = a^* a^*$  . (End of Proof.)

Theorem 8  $a \leq a^*$

Proof true

= {Theorem 6, unfolding once}

$$a^* = 1 \parallel a (1 \parallel a a^*)$$

= {Axiom 4 and Axiom 6}

$$a^* = 1 \parallel a \parallel a a a^*$$

$\Rightarrow$  {Axiom 1}

$$a^* \parallel a = a^*$$

(End of Proof.)

Theorem 9 Closure is monotonic, i.e.  $a \leq b \Rightarrow a^* \leq b^*$ .

Proof Under the assumption  $a \leq b$ , i.e.  $a \parallel b = b$ , we have to prove  $a^* \parallel b^* = b^*$ . Since  $b^* \leq a^* \parallel b^*$  is obvious (from Axiom 1), it suffices — on account of Theorem 0, antisymmetry — to prove  $a^* \parallel b^* \leq b^*$  under the assumption  $a \parallel b = b$ . We observe for any  $c$

$$c = a^* \parallel b^*$$

$\Rightarrow$  {Theorem 8}

$$c \leq (a^* \parallel b^*)^*$$

= {Axiom 7, with  $a := a^*$  and  $b := b^*$ }

$$c \leq (a^{**} b^{**})^*$$

= {Theorem 4}

$$c \leq (a^* b^*)^*$$

= {Axiom 7}

$$c \leq (a \parallel b)^*$$

= {assumption  $a \parallel b = b$ }

$c \leq b^*$  . (End of Proof.)

(The above proof is due to Rudolf H. Mak and Stefan Rönn.)

Theorem 10  $a \leq b^* \equiv a^* \leq b^*$  .

Proof.  $a \leq b^*$   $a^* \leq b^*$   
 $\Rightarrow \{ \text{Theorem 9} \}$   $\Rightarrow \{ \text{Theorem 8} \}$   
 $a^* \leq b^{**}$   $a \leq b^*$   
 $= \{ \text{Theorem 4} \}$   
 $a^* \leq b^*$  . (End of Proof.)

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The relation  $c \text{ from } (a,b)$  between an expression  $c$  and a pair of variables  $(a,b)$  is defined as the strongest relation satisfying

$0 \text{ from } (a,b)$

$a \text{ from } (a,b)$  and  $b \text{ from } (a,b)$

$(c \parallel d) \text{ from } (a,b) \equiv c \text{ from } (a,b) \wedge d \text{ from } (a,b)$

$(c d) \text{ from } (a,b) \equiv c \text{ from } (a,b) \wedge d \text{ from } (a,b)$

$c^* \text{ from } (a,b) \equiv c \text{ from } (a,b)$  .

It allows us to formulate

Theorem 11  $c \text{ from } (a,b) \Rightarrow c \leq (a \parallel b)^*$

Proof. The proof is by induction over the syntax. We observe for the base - mainly on account of Theorem 8 -  
 $0 \leq (a \parallel b)^*$  ,  $a \leq (a \parallel b)^*$  ,  $b \leq (a \parallel b)^*$  .

For the induction step, we prove under the hypothesis

$$c \leq (a \parallel b)^* \wedge d \leq (a \parallel b)^*$$

(i)  $c \parallel d \leq (a \parallel b)^*$  (Theorem 1 and Axiom 1, idempotence)

(ii)  $cd \leq (a \parallel b)^*$  (Corollary 0 and Theorem 7.)

(iii)  $c^* \leq (a \parallel b)^*$  (Theorem 10.)

(End of Proof.)

Combining Theorems 10 and 11 we get

Corollary 1  $c \text{ from } (a, b) \Rightarrow c^* \leq (a \parallel b)^*$

From Corollary 1, Theorem 9, and Theorem 0, antisymmetry, we conclude

Theorem 12  $a \parallel b \leq c \wedge c \text{ from } (a, b) \Rightarrow c^* = (a \parallel b)^*$ .

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Intermezzo In the mean time we have discovered a few improvements of the above. We should add

Theorem 1/3  $a \leq a \parallel b$

Proof true

= {Axiom 1}

$$a \parallel a \parallel b = a \parallel b$$

= {definition of  $\leq$ }

$$a \leq a \parallel b$$

(End of Proof.)

Theorem 2/3  $a \parallel b \leq c \equiv a \leq c \wedge b \leq c$

Proof  $a \parallel b \leq c$

= {Theorem 1/3}

$$a \leq a \parallel b \wedge b \leq a \parallel b \wedge a \parallel b \leq c$$

$$\Rightarrow \{ \text{Theorem 0, transitivity} \}$$

$$a \leq c \wedge b \leq c$$

$$a \leq c \wedge b \leq c$$

$$= \{ \text{definition of } \leq \}$$

$$a \parallel c = c \wedge b \parallel c = c$$

$$\Rightarrow \{ \text{Leibniz} \}$$

$$a \parallel (b \parallel c) = c$$

$$= \{ \text{definition of } \leq \}$$

$$a \parallel b \leq c \quad . \quad (\text{End of Proof.})$$

In connection with Axiom 2 it would have been appropriate to recall the general

Theorem For a binary operator with a right and a left unit element, the unit element is unique.

Proof ( $\exists a, b :: UL \ \$ \ a = a \wedge b \ \$ \ UR = b$ )

$$\Rightarrow \{ a := UR ; b := UL \}$$

$$UL \ \$ \ UR = UR \wedge UL \ \$ \ UR = UL$$

$$\Rightarrow \{ \text{Leibniz} \}$$

$$UL = UR \quad . \quad (\text{End of Proof.})$$

The theorem is of equal relevance for Axiom 6.

We are tempted to replace Axiom 5 by

Axiom 5'  $ab = 0 \equiv a = 0 \vee b = 0$  .

The original Axiom 5 would then get the status of a corollary.

In a similar vein we are tempted to strengthen Axiom 6 by adding to it

$$\underline{\text{Axiom 6}'/2} \quad 1 \leq ab \equiv 1 \leq a \wedge 1 \leq b$$

$$\underline{\text{Theorem 2}'/2} \quad 1 = ab \equiv 1 = a \wedge 1 = b$$

$$\underline{\text{Proof}} \quad 1 = a \wedge 1 = b$$

$$\Rightarrow \{\text{Axiom 6}\}$$

$$1 = ab$$

$$1 = ab$$

$$= \{\text{Theorem 0, reflexivity}\}$$

$$1 \leq ab \wedge 1 = ab$$

$$= \{\text{Axiom 6}'/2\}$$

$$1 \leq a \wedge 1 \leq b \wedge 1 = ab$$

$$= \{\text{monotonicity of concatenation and Axiom 6}\}$$

$$1 \leq a \wedge 1 \leq b \wedge 1 = ab \wedge b \leq ab \wedge a \leq ab$$

$$\Rightarrow \{\text{Leibniz}\}$$

$$1 \leq a \wedge 1 \leq b \wedge b \leq 1 \wedge a \leq 1$$

$$= \{\text{Theorem 0, antisymmetry}\}$$

$$1 = a \wedge 1 = b \quad (\text{End of Proof.})$$

We probably need as well

$$\underline{\text{Axiom vi}} \quad 1 \leq a \parallel b \equiv 1 \leq a \vee 1 \leq b$$

$$\underline{\text{Axiom vii}} \quad a \leq 1 \equiv a = 0 \neq a = 1$$

(With  $a := 1$ , we derive from Axiom vii  $1 \neq 0$ .)

(End of Intermezzo.)



$$(aa)^* \parallel a(aa)^* = a^* \quad (1)$$

We could try to solve the problem by postulating that (1) has a unique solution, but that would be a mistake, as is shown by the following analysis.

Theorem 13  $1 \leq a \equiv a^* = aa^*$

Proof  $a^* = aa^*$

$$= \{ \text{Theorem 6} \}$$

$$1 \parallel aa^* = aa^*$$

$$= \{ \text{definition of } \leq \}$$

$$1 \leq aa^*$$

$$= \{ \text{Axiom 6}'/2 \}$$

$$1 \leq a \wedge 1 \leq a^*$$

$$= \{ 1 \leq a^* \}$$

$$1 \leq a$$

(End of Proof.)

We now show that for  $1 \leq a$  equation (1) is solved by  $a^* \parallel a^*c$ . To this end we observe

$$z = 1 \parallel a(a^* \parallel a^*c)$$

$$= \{ \text{Axiom 4} \}$$

$$z = 1 \parallel aa^* \parallel a^*c$$

$$= \{ \text{Theorem 6} \}$$

$$z = a^* \parallel a^*c$$

$$= \{ 1 \leq a \text{ and Theorem 13} \}$$

$$z = a^* \parallel a^*c$$

Note that in the above observation we have not made any assumption about  $c$ . Since  $1 \leq 1$ ,

the above observation with  $a:=1$  and  $1^*=1$  — Theorem 5 — tells us that the postulate that (0) has a unique solution leads to the conclusion that  $1 \parallel c$  is unique, i.e. independent of  $c$ . From the idempotence of  $\parallel$  we would then conclude  $1 \parallel c = 1$ , i.e.  $c \leq 1$  for any  $c$ . Axiom vii then tells us that for any  $c$  we have  $c=0 \vee c=1$ . Such a universe, however, is too meagre to our taste: in view of our earlier remark that 0 and 1 differ from all the letters and — Axiom 0 — that each letter is a regular expression, we would be considering the not so interesting situation of an empty alphabet.