Transitivity as an (unexpected?) consequence

All variables in this note are of an unspecified type on which a relation $\leq$ is defined.

A standard technique for function definition is to define the function as the solution of an equation. We would like to define the solution of

(0) $\quad w : \langle \forall z :: w \leq z \equiv x \leq z \wedge y \leq z \rangle$

as function of x and y. This only works provided

(i) solutions of (0) are always unique, and
(ii) solutions of (1) always exist.

To begin with, we investigate what to postulate about $\leq$ so that we can prove unicity. Let a, b be solutions of (0); we then observe for arbitrary x, y

$$
\begin{array}{ll}
 & true \\
= & \quad \{\text{pred.calc.}\} \\
 & \langle \forall z :: x \leq z \wedge y \leq z \equiv x \leq z \wedge y \leq z \rangle \\
= & \quad \{\, a \text{ and } b \text{ both solve } (0) \} \\
 & \langle \forall z :: a \leq z \equiv b \leq z \rangle \\
\Rightarrow & \quad \{\text{instantiate with } z := a \text{ and with } z := b\} \\
 & (a \leq a \equiv b \leq a) \wedge (a \leq b \equiv b \leq b) \\
= & \quad \{\bullet \text{ assume } \leq \text{ to be reflexive}\}
\end{array}
$$

$$b \leq a \ \wedge \ a \leq b$$
$$\Rightarrow \quad \{\bullet \text{ assume } \leq \text{ to be antisymmetric}\}$$
$$a = b \quad .$$

In view of the above argument we postulate our $\leq$ to be reflexive and antisymmetric.

<u>Remark</u> $\leq$ being both reflexive and antisymmetric is expressed by the single formula

(1) $\quad [x = y \ \equiv \ x \leq y \wedge y \leq x] \quad ,$

where we use $[\ldots]$ to denote universal quantification of the enclosed expression over its free variables. As the above argument shows, (1) implies

(2) $\quad [x = y \ \equiv \ \langle \forall z :: \ x \leq z \ \equiv \ y \leq z \rangle] \quad$ and

(3) $\quad [x = y \ \equiv \ \langle \forall z :: \ z \leq x \ \equiv \ z \leq y \rangle] \quad .$

(End of Remark.)

Besides (1) we now postulate (ii), viz. that (0) is always solvable. The rest of this note is devoted to the proof that $(1) \wedge (ii)$ implies the transitivity of $\leq$.

* * *

The proof, totally elementary and built from well-known components – see, for instance, [0] –

is rather indirect: following Feijen, we use (0)'s unique solution to define an infix operator, of which we prove a number of properties using (1) and (2). Finally, the transitivity of $\leq$ is shown. Here we go.

We denote (0)'s solution by $x\uparrow y$, i.e. define the infix operator $\uparrow$ by

(4) $\qquad [x\uparrow y \leq z \;\equiv\; x\leq z \wedge y\leq z]$.

<u>Lemma</u> 0: $\uparrow$ is associative.

<u>Proof</u>
$$
\begin{array}{ll}
 & (a\uparrow b)\uparrow c \leq z \\
= & \quad \{ (4);\ x,y := (a\uparrow b), c \} \\
 & (a\uparrow b) \leq z \ \wedge\ c\leq z \\
= & \quad \{ (4);\ x,y := a,b \} \\
 & (a\leq z \wedge b\leq z) \ \wedge\ c\leq z \\
= & \quad \{ \wedge \text{ is associative} \} \\
 & a\leq z \ \wedge\ (b\leq z \wedge c\leq z) \\
= & \quad \{ (4):\ x,y := b,c \} \\
 & a\leq z \ \wedge\ (b\uparrow c)\leq z \\
= & \quad \{ (4):\ x,y := a, (b\uparrow c) \} \\
 & a\uparrow(b\uparrow c) \leq z \qquad ;
\end{array}
$$

on account of (2)

$$[(a\uparrow b)\uparrow c = a\uparrow (b\uparrow c)]$$

now follows. (End of Proof)

<u>Lemma 1</u>: $[x\leq x\uparrow y \ \wedge\ y\leq x\uparrow y]$.

<u>Proof</u> We observe for any x, y

$$
\begin{array}{ll}
 & x \leq x\uparrow y \ \wedge \ y \leq x\uparrow y \\
= & \quad \{ (4):\ z := x\uparrow y \} \\
 & x\uparrow y \leq x\uparrow y \\
= & \quad \{ \leq \text{ reflexive} \} \\
 & \text{true}
\end{array}
$$

(End of Proof)

<u>Lemma</u> 2 $[x\uparrow y = y \ \equiv \ x \leq y]$.

<u>Proof</u> We observe for any x, y

$$
\begin{array}{ll}
 & x\uparrow y = y \\
= & \quad \{ (1):\ x := x\uparrow y \} \\
 & x\uparrow y \leq y \ \wedge \ y \leq x\uparrow y \\
= & \quad \{ \text{Lemma } 1 \} \\
 & x\uparrow y \leq y \\
= & \quad \{ (4):\ z := y \} \\
 & x \leq y \ \wedge \ y \leq y \\
= & \quad \{ \leq \text{ reflexive} \} \\
 & x \leq y \quad .
\end{array}
$$

(End of Proof)

<u>Theorem:</u> $\leq$ is transitive .

<u>Proof</u> We observe for any a, b, c

$$
\begin{array}{ll}
 & a \leq b \ \wedge \ b \leq c \\
= & \quad \{ \text{Lemma } 2:\ x, y := a, b \text{ and } x, y := b, c \} \\
 & a\uparrow b = b \ \wedge \ b\uparrow c = c \\
\Rightarrow & \quad \{ \text{Leibniz} \} \\
 & (a\uparrow b)\uparrow c = c \ \wedge \ b\uparrow c = c \\
= & \quad \{ \text{Lemma } 0 \}
\end{array}
$$

$$
\begin{array}{ll}
 & a\uparrow(b\uparrow c) = c \;\wedge\; b\uparrow c = c \\
\Rightarrow & \quad \{\text{Leibniz}\} \\
 & a\uparrow c = c \\
= & \quad \{\text{Lemma 2: } x,y := a,c\} \\
 & a \leq c \quad .
\end{array}
$$

(End of Proof.)

*   *   *

The proofs have been included to make this note self-contained (and perhaps also because it is always a pleasure to write them down). It is instructive to check how essential use has been made of equality. (Lemma 2, replaced, for instance, by $[x\uparrow y \leq y \equiv x \leq y]$ or $[x\uparrow y = y \Leftarrow x \leq y]$, would not have sufficed for the proof of the Theorem.]

[0] W.H.J. Feijen, "Exercises in Formula Manipulation" in "Formal Development of Programs and Proofs" (Edsger W. Dijkstra, Ed.), Addison-Wesley Publishing Company, Inc., 1990

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