A simple country dance for transitivity and shunting

Transitivity of the relation $\sqsubseteq$ (read "under") is traditionally expressed by stating that for all a, b, c

(0) $\quad a \sqsubseteq b \wedge b \sqsubseteq c \Rightarrow a \sqsubseteq c$.

With the converse relation $\sqsupseteq$ (read "above") satisfying $a \sqsubseteq b \equiv b \sqsupseteq a$ etc., and the symmetry of $\wedge$, we can rewrite (0) and reformulate the transitivity by stating that for all a, b, c

(1) $\quad c \sqsupseteq b \wedge b \sqsupseteq a \Rightarrow c \sqsupseteq a$,

in other words: of $\sqsubseteq$ and $\sqsupseteq$, the one is as transitive as the other.

But in this note we shall exploit the symmetry differently, viz by observing that there are two ways of rewriting (0) by shunting, viz.

(2) $\quad b \sqsubseteq c \Rightarrow (a \sqsubseteq b \Rightarrow a \sqsubseteq c)$

(3) $\quad a \sqsubseteq b \Rightarrow (b \sqsubseteq c \Rightarrow a \sqsubseteq c)$,

and if we take for $\sqsubseteq$ the implication,

we get

(4) $(b \Rightarrow c) \Rightarrow ((a \Rightarrow b) \Rightarrow (a \Rightarrow c))$

(5) $(a \Rightarrow b) \Rightarrow ((b \Rightarrow c) \Rightarrow (a \Rightarrow c))$,

i.e. implication is monotonic in its right-hand argument and antimonotonic in the other.

Here, however, we shall pursue that in (0), (2), and (3), one of the three dummies in the universal quantification occurs only at one side of the (outer) implication. Universally quantifying (0), (2), (3) over b, a, c respectively, yields secundum regulas artis

(0') $\langle \exists b :: a \sqsubseteq b \wedge b \sqsubseteq c \rangle \Rightarrow a \sqsubseteq c$

(2') $b \sqsubseteq c \Rightarrow \langle \forall a :: a \sqsubseteq b \Rightarrow a \sqsubseteq c \rangle$

(3') $a \sqsubseteq b \Rightarrow \langle \forall c :: b \sqsubseteq c \Rightarrow a \sqsubseteq c \rangle$.

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This note is written because I observed people having difficulty with the first step of a reflexivity proof that started with

$$
\begin{array}{ll}
 & a \sqsubseteq a \\
\Leftarrow & \quad \{\ (5, \mathbb{D}) \text{ with } c := a\ \} \\
 & \langle \exists b ::\ a \sqsubseteq b \rightarrowtail a \ \wedge\ a \sqsubseteq b \rangle
\end{array}
$$

[where $(5, \mathbb{D})$ in the hint referred to

$$a \sqsubseteq b \rightarrowtail c \ \wedge\ a \sqsubseteq b \ \Rightarrow\ a \sqsubseteq c\ ]\ .$$

People wonder: "Why that existential quantification?" Why not just write

$$
\begin{array}{ll}
 & a \sqsubseteq a \\
\Leftarrow & \quad \{\ (5, \mathbb{D}) \text{ with } c := a\ \} \\
 & a \sqsubseteq b \rightarrowtail a \ \wedge\ a \sqsubseteq b \qquad\qquad ?
\end{array}
$$

But that would be a step that transforms an expression not depending on b into one depending on b. We can eliminate b as free variable by quantifying over it, and I used to justify the choice of the existential quantification by the fact that this proof is building up a strengthening chain and that therefore the weaker intermediate result makes it easier to complete it. (Remember: in a strengthening chain, each intermediate result represents your remaining proof obligation.)

Similarly, if in a weakening chain

a free variable threatens to be introduced we bind it by universal quantifications, because in a weakening chain, intermediate results stand for conclusions reached so far, and one wishes to formulate those as strong as possible.

It is these strategies that are nicely reflected in (0'), (2') and (3').

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