Lecture 9: Geometric Modeling and Visualization

Geometric Partial Differential Equations:
Non-Linear Surface & Volume Diffusion

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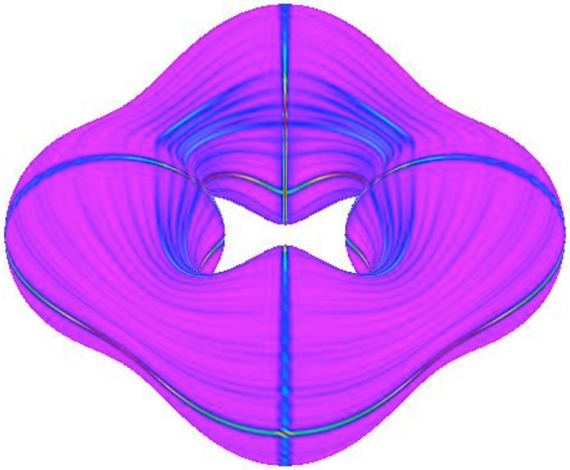
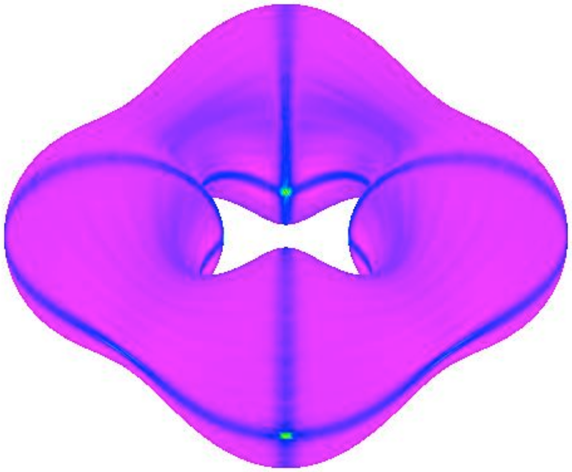
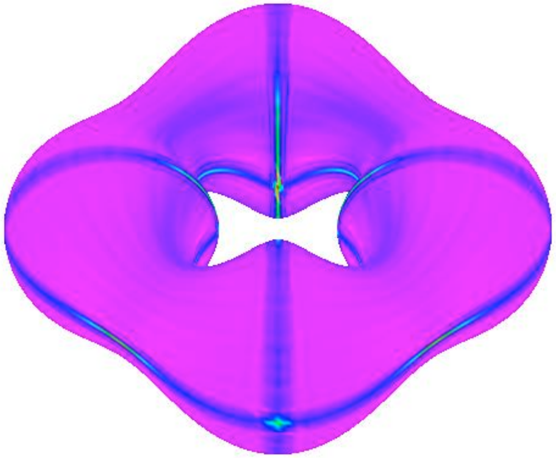
Fairing Noisy Surfaces (Mean Curvature)

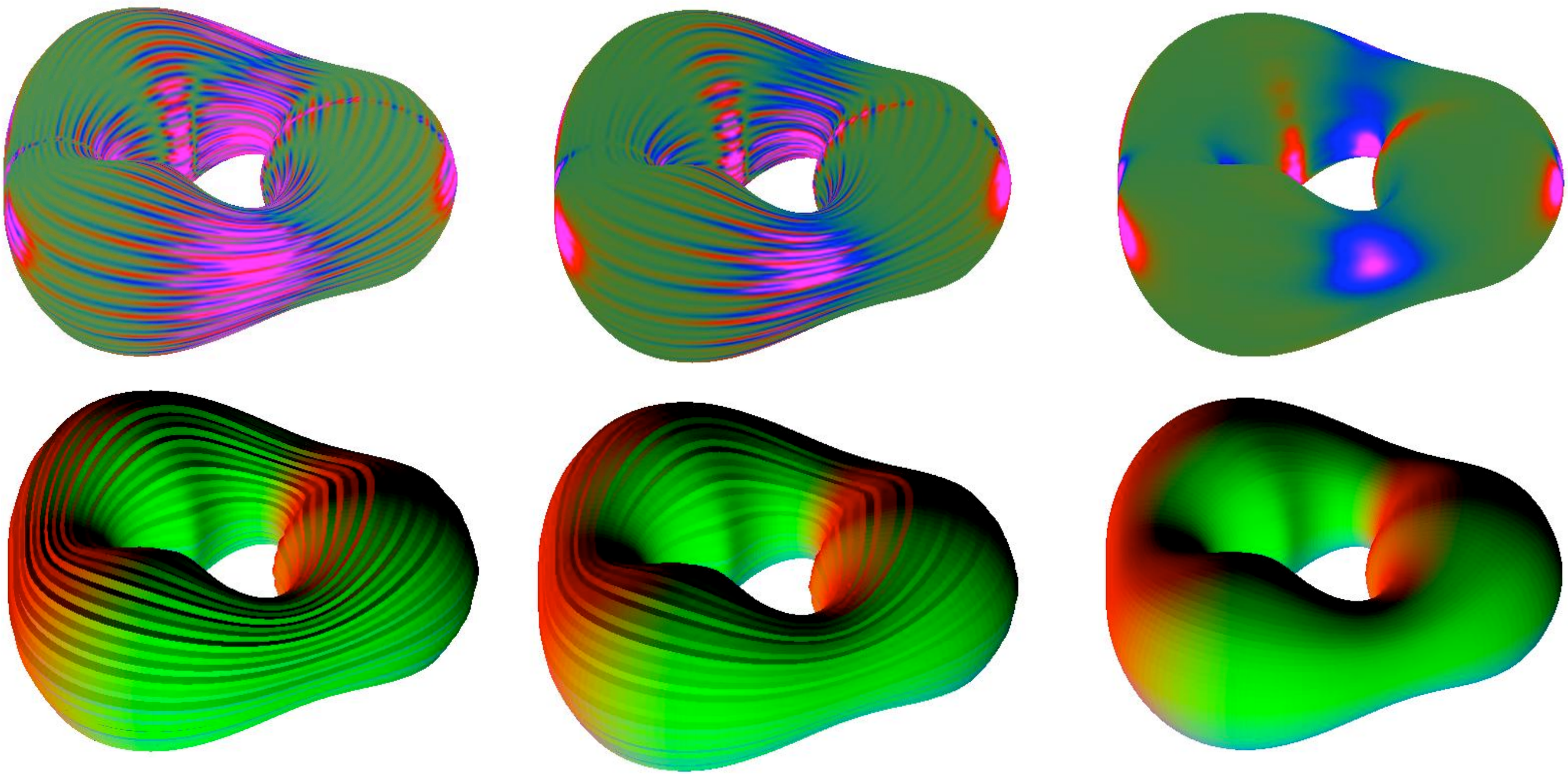

Initial functions


After three iterations


After five iterations

Mean curvature plot: non-smooth functions at x=0, y=0, z=0

Fairing Noisy Surfaces (Gaussian Curvature)

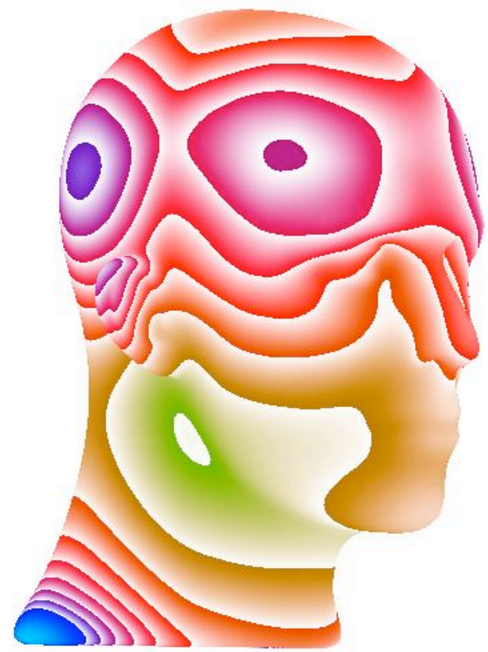


Initial data | After 1 iteration | After 4 iterations

Fairing of Scalar Function on Surface

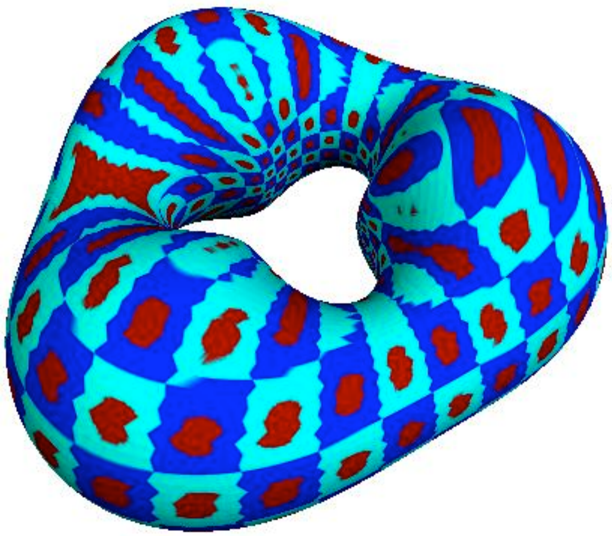
Iso – Contours of Acoustic Amplitude

Initial data

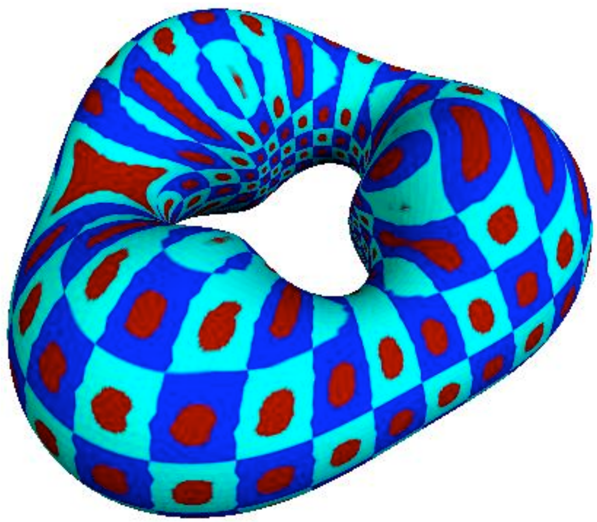


After 4 fairing iterations

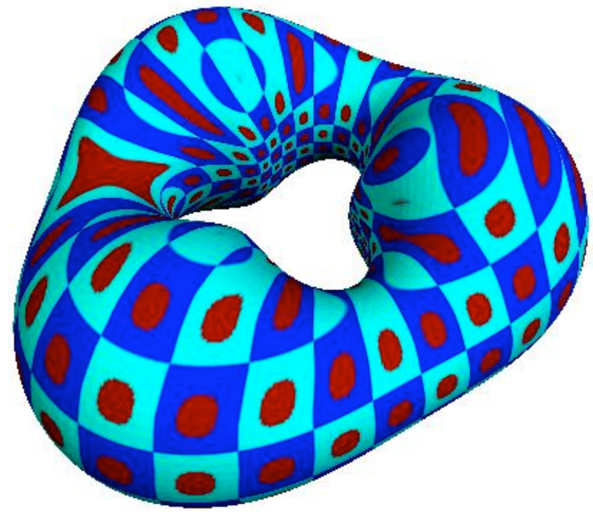
Functions on Surface: Texture



Initial dada



After 1 iteration



After 4 iterations

Problem Considered

Given a discretized noisy triangular surface mesh $\boldsymbol{G_d} \subset \boldsymbol{R}^3$ (geometric information) and a discretized noisy function-vector $\mathbb{F}_d$.

Our goals are :

- Smooth out the noise and to obtain smooth geometry as well as surface function data at different scales.
- Construct continuous (non-discretized) representations for the smoothed geometry and surface function data.
- Provide approaches for visualizing the smoothness of both the geometric and physical information during the smoothing process.

Related work in Image Processing

- Gabor ,1965, PDE based image processing, Jian, 1977, Took off thanks to Koenderink, 1984 Witkin 1983.

- Perona and Malik, 1990, anisotropic diffusion, smoothing and enhancing sharp features.

- Osher and Sethian, 1988, curvature based velocities.

- Mumford and Shah, 1989, PDE based segmentation.

- Terzopoulos et al, 1988, PDE based on active contours for image segmentation.

Previous Work for Mesh Fairing

1. Optimization

   a. Minimize thin plate energy (Kobbelt 1996, Desbrun, Meyer, Schroder, 1999).

$$E_p(f) = \int f_{uu}^2 + 2f_{uv}^2 + f_{vv}^2$$

   b. Minimize membrane energy(Kobbelt, 1998 , Desbrun, Meyer, Schroder, 1999).

$$E_m(f) = \int f_u^2 + f_v^2$$

   c. Minimize curvature (Welch, Witkin, 1992).

$$E_c(S) = \int \kappa_1^2 + \kappa_2^2$$

   d. Spring energy(2000).

Previous Work for Mesh Fairing

2. Signal Processing(Guskov, Sweldens, Schroder,1999; Taubin, 1995) using surface relaxation as low pass filter

$$Rp_i = \sum_{j \in V_2(i)} w_{i,j} p_j$$

where $w_{i,j}$ are chosen to minimize something, e.g. the dihedral angles.

Geometry Driven Diffusion

Evolution (time dependent)

Linear heat conduction equation.

$$\partial_t \rho - \Delta \rho = 0 , \qquad \Delta = \mathrm{div} \cdot \nabla$$

For equalizing spatial variation in concentration

Geometry Driven Diffusion

For the surface *M*, the counterpart of the Laplacian Δ is the Laplace Beltrami operator Δ_M. Hence, one obtains the geometric diffusion equation

$$\partial_t x - \Delta_M x = 0, \qquad \Delta_M = \mathrm{div}_M \cdot \nabla$$

for surface point *x(t)* on the surface *M(t)*

Model of Geometric Diffusion

Partial Differential Equation

$$\partial_t x(t) - \mathrm{div}_{M(t)}(\nabla_{M(t)} x(t)) = 0$$

$$M(0) = M$$

where M(t) is the solution surface at time t, x(t) is surface point.

Divergence $\mathrm{div}_{M(t)}\ \boldsymbol{v}$ for a vector field $\boldsymbol{v} \in V$ is defined as the dual operator of the gradient:

$$\int_M \mathrm{div}_M v\phi dx := -\int_M v^T \nabla \phi dx, \quad \forall \phi \in C^\infty(M)$$

Variational form

$$\left(\partial_t x(t), \theta\right)_{M(t)} + \left(\nabla_{M(t)} x(t), \nabla_{M(t)} \theta\right)_{TM(t)} = 0,$$

$$\forall \theta \in C^{\infty}(M(t))$$

where

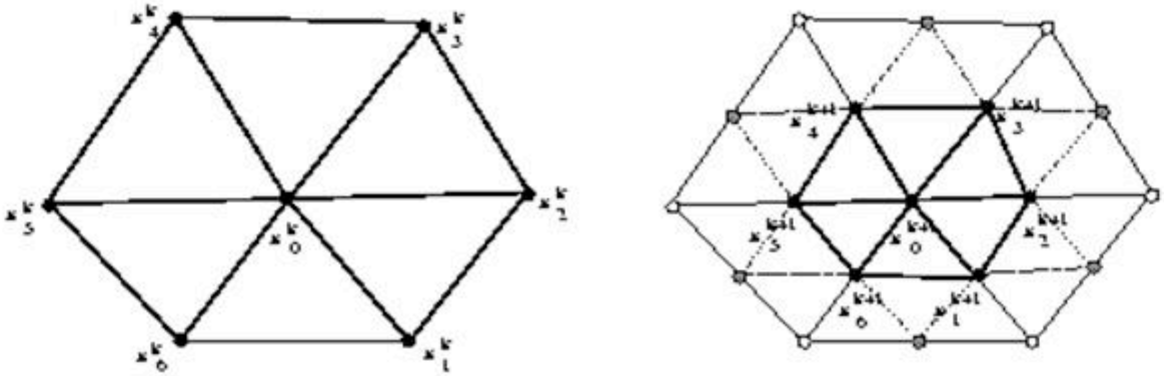
$$(f, g)_M = \int_M fg\,dx, \qquad (\phi, \psi)_{TM} = \int_M \phi^T \psi\,dx$$

- How to represent $M(t)$?
- How to choose θ ?

Loop's Subdivision Surface

Edge rule:

$$x_i^{k+1} = \frac{3x_0^k + 3x_i^k + x_{i-1}^k + x_{i+1}^k}{8}, i = 1, \ldots, n,$$

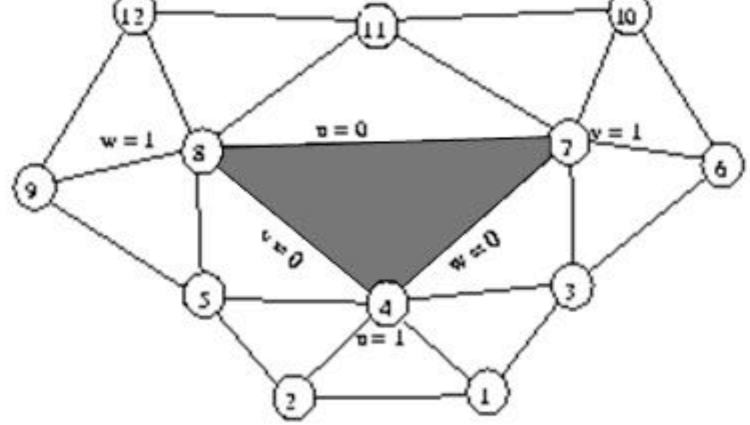


Refinement of a triangular mesh around a vertex

Vertex rule:

$$x_0^{k+1} = (1 - na)x_0^k + a(x_1^k + x_2^k + \ldots + x_n^k).$$

Limit Surface – Regular Case



$$N_1 = \tfrac{1}{12}(u^4 + 2u^3 v),$$

$$N_2 = \tfrac{1}{12}(u^4 + 2u^3 w),$$

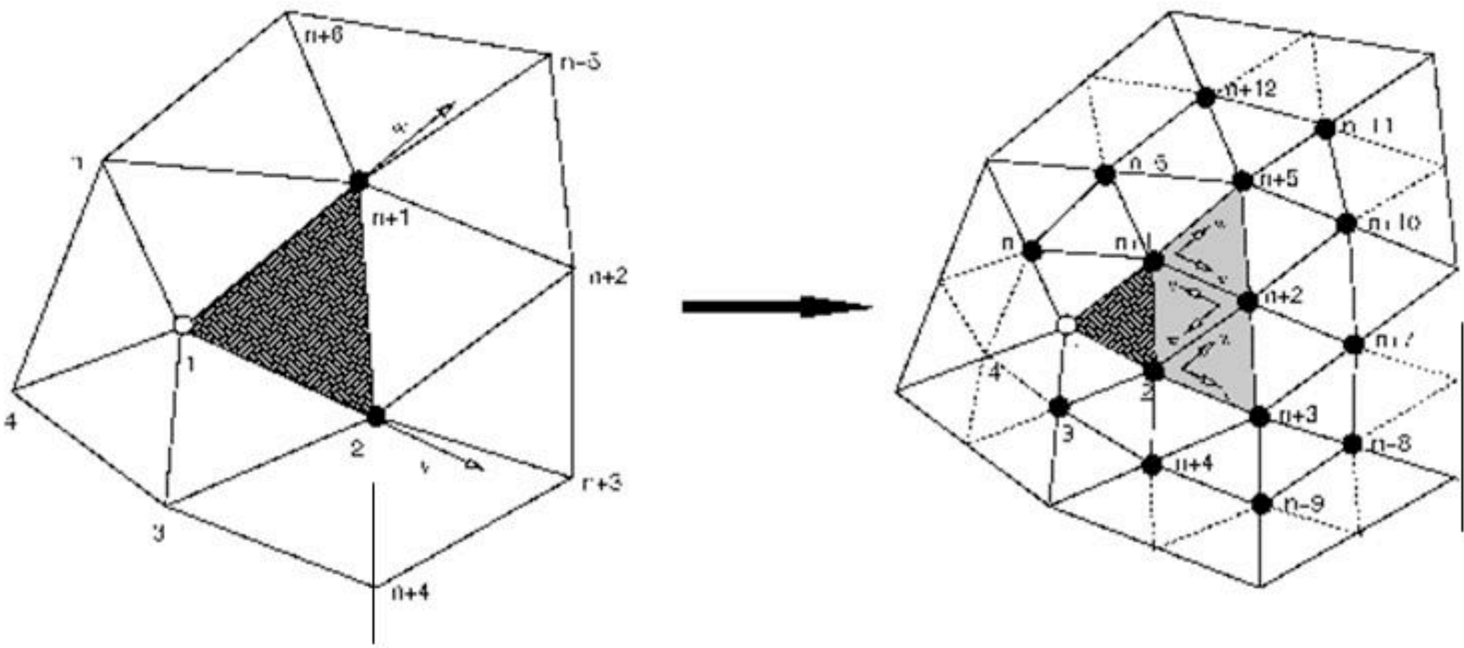
$$N_3 = \tfrac{1}{12}[u^4 + v^4 + 6u^3 v + 6uv^3 + 12u^2 v^2 + (2u^3 + 2v^3 + 6u^2 v + 6uv^2)w],$$

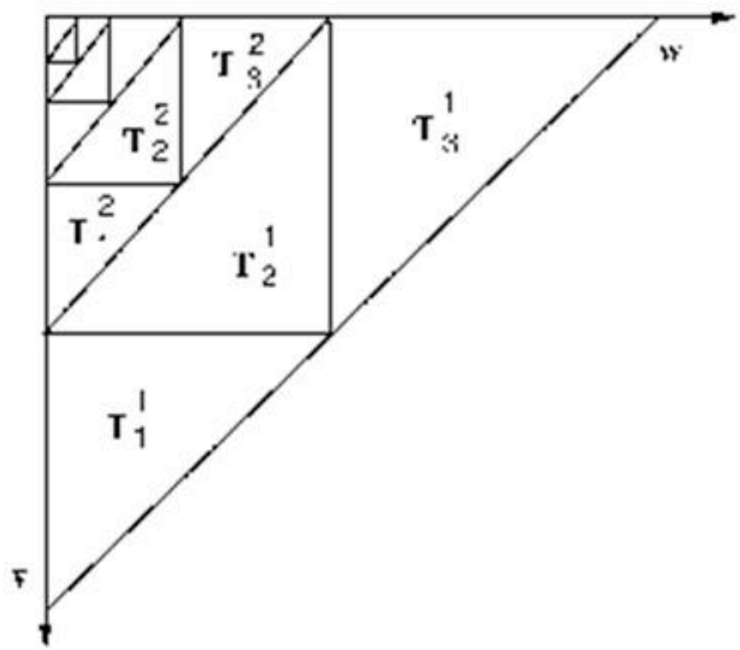
$$N_4 = \tfrac{1}{12}[6u^4 + 24u^3(v+w) + u^2(24v^2 + 60vw + 24w^2) + u(8v^3 + 36v^2 w + 36vw^2 + 8w^3) + (v^4 + 6v^3 w + 12v^2 w^2 + 6vw^3 + w^4)]$$

$$(u, v, w) \rightarrow (v, w, u): \quad N_1, N_2, N_3, N_4 \rightarrow N_{10}, N_6, N_{11}, N_7$$

$$(u, v, w) \rightarrow (w, u, v): \quad N_1, N_2, N_3, N_4 \rightarrow N_9, N_{12}, N_5, N_8$$

Limit Surface – Irregular Case





Refinement in the parametric space

Here the main task is to compute the new control vertices. As usual, the subdivision around an irregular patch is formulated as a linear transform from the level $(k\text{-}1)$ 1-ring vertices of the irregular patch to the related level k vertices, i.e.,

$$X^k = AX^{k-1} = \ldots = A^k X^0,$$

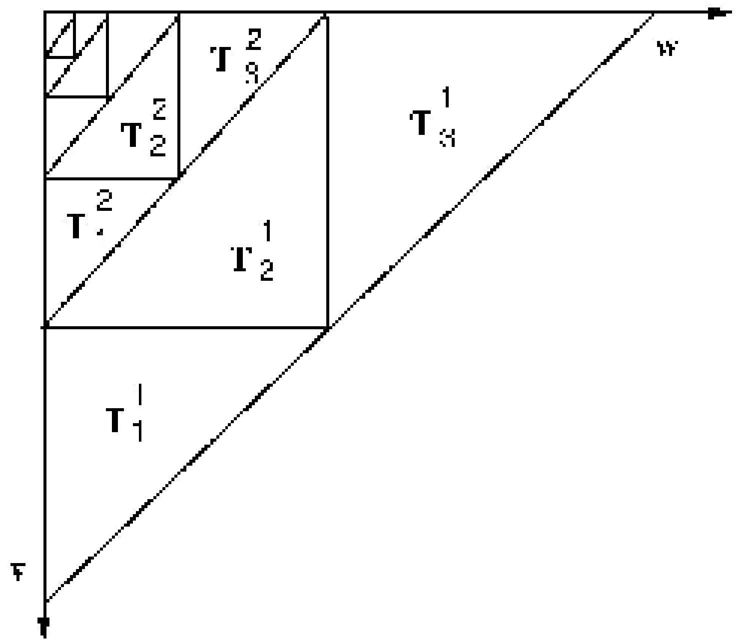
$$\widetilde{X}^{k+1} = \widetilde{A} X^k = \widetilde{A} A^k X^0,$$

where

$$X^k = [\boldsymbol{x}_1^k, \ldots, \boldsymbol{x}_{n+6}^k]^T, \quad \widetilde{X}^k = [\boldsymbol{x}_1^k, \ldots, \boldsymbol{x}_{n+6}^k, \boldsymbol{x}_{n+7}^k, \ldots, \boldsymbol{x}_{n+12}^k]^T$$

Using Jordan canonical form

$$A = TJT^{-1}, \quad A^k = TJ^kT^{-1}$$



Refinement in the parametric space

Spatial Discretization

$$(\partial_t \boldsymbol{x}(t), \theta)_{M(t)} + \left(\nabla_{M(t)} \boldsymbol{x}(t), \nabla_{M(t)} \theta\right)_{TM(t)} = 0, \quad \forall \boldsymbol{\theta} \in V_{M(t)}$$

Let

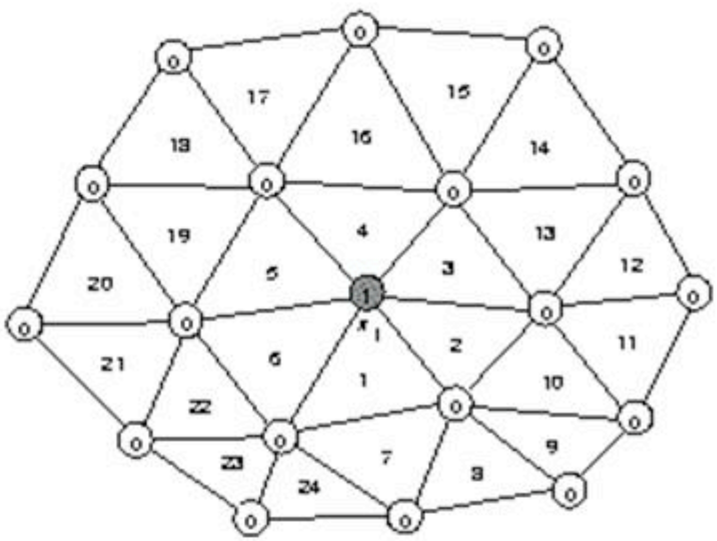
$$\boldsymbol{x}(t) = \sum_{i=1}^{m} c_i(t) \phi_i(\boldsymbol{x}), \quad \boldsymbol{\theta} = \phi_j(\boldsymbol{x})$$

Then we have a set of ordinary differential equations

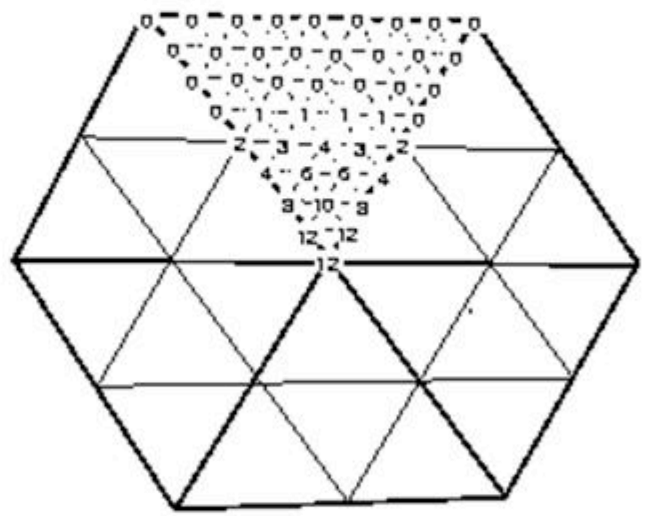
$$\sum_{i=1}^{m} c_i'(t) (\phi_i(x), \phi_j(x))_{M(t)} + \sum_{i=1}^{m} c_i(t) (\nabla_{M(t)} \phi_i(\boldsymbol{x}), \nabla_{M(t)} \phi_j(\boldsymbol{x}))_{TM(t)} = 0$$

$$j = 1, \cdots, m$$

Where ϕ_i are the basis functions



(a)



(b)

Time Discretization

Let X^n be approximation of $x(n\tau)$, where τ is the timestep. Then
The semi-implicit discretization is

$$\left(\frac{X^{n+1}-X^n}{\tau}, \phi_i\right)_{M(n\tau)} +$$

$$(\nabla_{M(n\tau)} X^{n+1}, \nabla_{M(n\tau)} \phi i)_{TM(n\tau)} = 0, i = 1, \cdots, m$$

Since

$$x(t) = \sum_{i=1}^{m} c_i(t)\phi_i(x)$$

Then we have a linear system.

$$(M^n + \tau L^n)C((n+1)\tau) = M^n C(n\tau)$$

where $C(t) = [c_1(t), \cdots, c_m(t)]$

$$M^n = \left((\phi_i, \phi_j)_{M(n\tau)}\right)_{i,j=1}^{m}$$

and

$$L^n = \left((\nabla_{M(n\tau)}\phi_i, \nabla_{M(n\tau)}\phi_j)_{TM(n\tau)}\right)_{i,j=1}^{m}$$

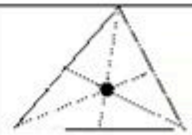
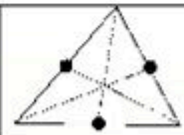
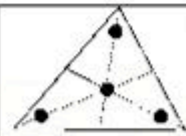
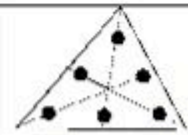
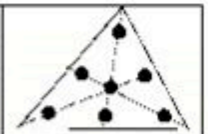
Solving the Linear System

- M^n and L^n are sparse.
- M^n is symmetric and positive definite.
- L^n is symmetric and nonnegative definite.
- $M^n + \tau L^n$ is symmetric and positive definite.

The system is solved by Gauss Seidel iteration or conjugate gradient method.

?1. What is the best approach?
?2. How to determine the optimal step length?

Numerical Integration

					
u_1	0.3333333333	0.0	0.1333333333	0.8168475729	0.05961587
u_2		0.5	0.1333333333	0.0915762135	0.47014206
u_3		0.5	0.7333333333	0.0915762135	0.47014206
u_4			0.3333333333	0.1081030181	0.79742699
u_5				0.4459484909	0.10128651
u_6				0.4459484909	0.10128651
u_7					0.33333333
w_1	0.3333333333	0.5	0.7333333333	0.0915762135	0.47014206
w_2		0.0	0.1333333333	0.8168475729	0.05961587
w_3		0.5	0.1333333333	0.0915762135	0.47014206
w_4			0.3333333333	0.4459484909	0.10128651
w_5				0.1081030181	0.79742699
w_6				0.4459484909	0.10128651
w_7					0.33333333
W_1	1.0	0.3333333333	0.5208333333	0.1099517436	0.13239415
W_2		0.3333333333	0.5208333333	0.1099517436	0.13239415
W_3		0.3333333333	0.5208333333	0.1099517436	0.13239415
W_4			0.5625	0.2233815896	0.12593918
W_5				0.2233815896	0.12593918
W_6				0.2233815896	0.12593918
W_7					0.225
p	1	2	3	4	5

Integration rules over triangle. $(1-\xi_i-\eta_i, \xi_i, \eta_i)$ are barycentric coordinates of the nodes, W_i are the weights. The last row represents the algebraic precision.

Anti-Shrinking

Denote the x,y and z components of the surface point $x(t)$ as $x_1(t)$, $x_2(t)$ and $x_3(t)$, respectively. Then, we have

$$(\partial_t x_i(t), x_i(t))_{M(t)} = -(\nabla_{M(t)} x_i(t), \nabla_{M(t)} x_i(t))_{TM(t)}$$

and

$$\frac{\partial(\boldsymbol{x}(t),\boldsymbol{x}(t))_{M(t)}}{\partial t} = 2(\partial_t x(t), x(t))_{M(t)} = -4Area(M(t))$$

$$\frac{\partial(Area(M(t))}{\partial t} = -\int_{M(t)} H^2 dx$$

Since $Area(M(t)) > 0$, the surface point $x(t)$ shrinks towards the origin at the average speed of $4Area(M(t))$.

Since

$$\Delta_M x = -H(x)N(x)$$

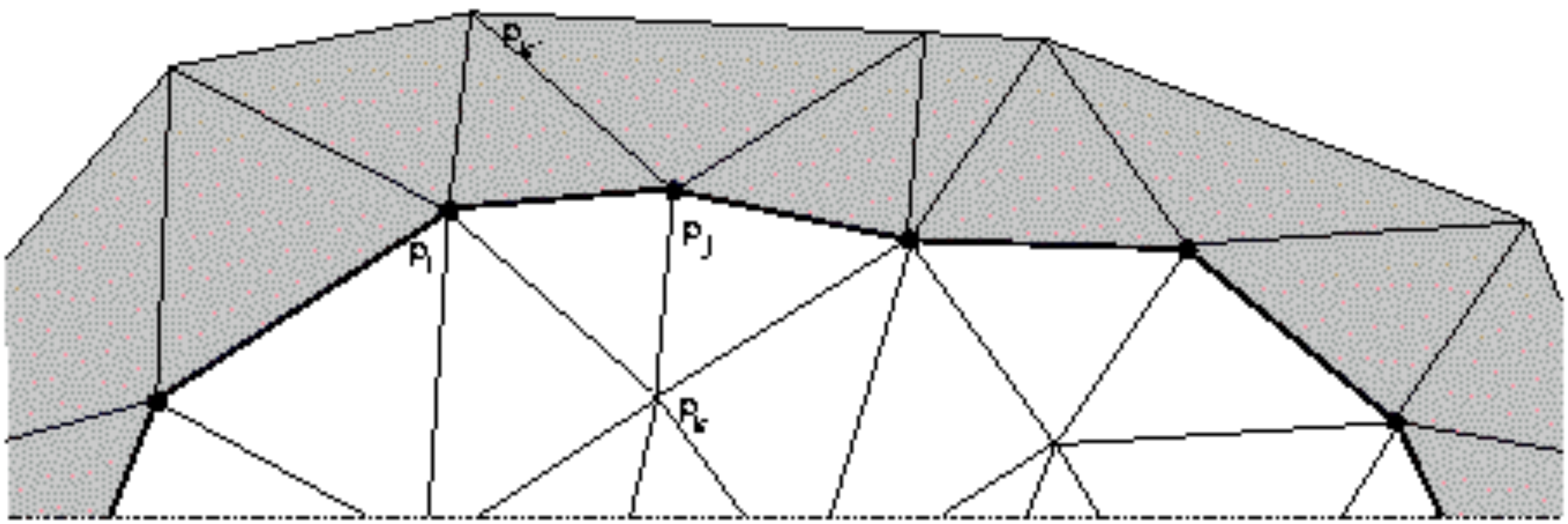
we have

$$\partial_t x = -H(x)N(x)$$

$$\tfrac{d}{dt}(x(t), x(t))_{M(t)} = -4Area(M(t))$$

$$\tfrac{d}{dt}Area(\omega(t)) = -\int_{\omega(t)} H^2 dx$$

Open Surface



Diffusion Tensor

$$\partial_t x(t) - div(a(x)\nabla_{M(t)} x(t)) = 0$$

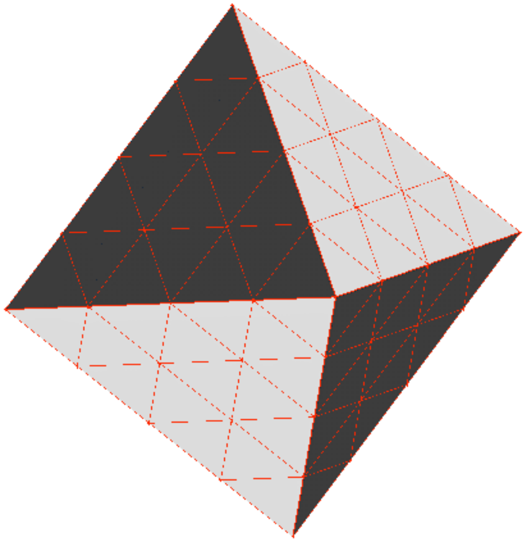
a(x) is a symmetric, positive define linear mapping on the Tangent space

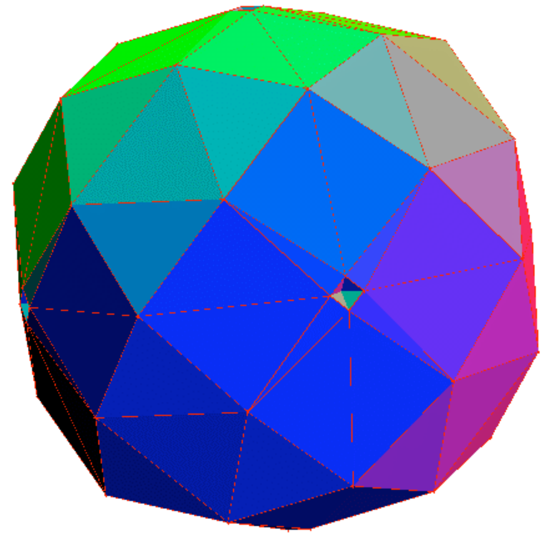
$$a(x) : TM \to TM$$

The problem is how to choose the diffusion tensor?

Anti – Crease by Diffusion Tensor

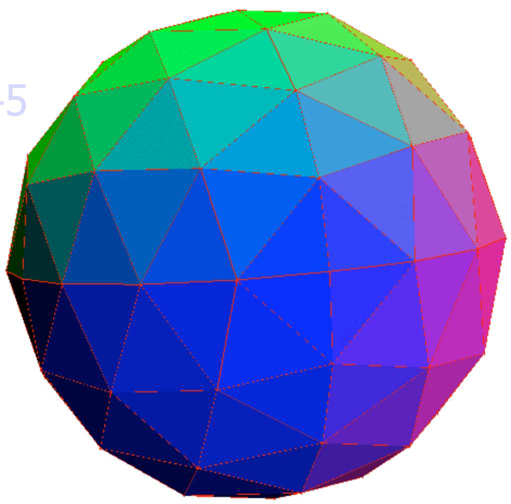
Initial
Mesh

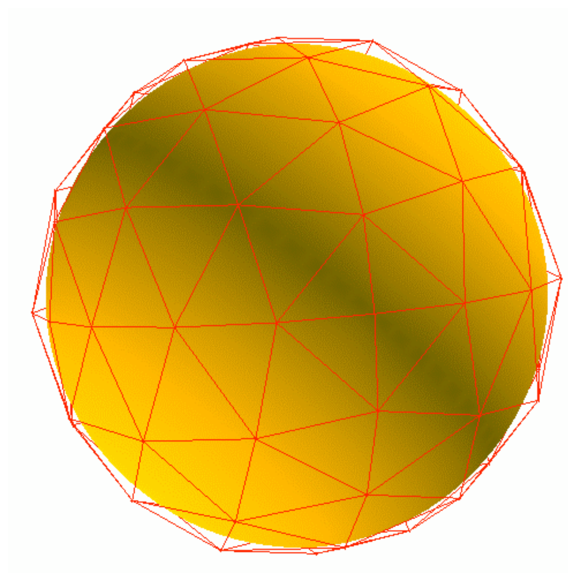




After
11,114
iterations

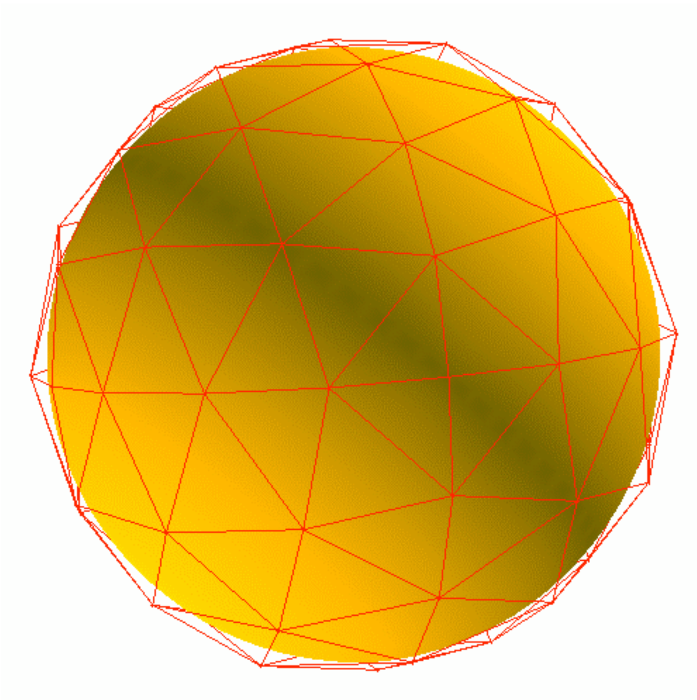
$a=(\text{area})^{0.45}$
of triangle.
After 2228
iterations.

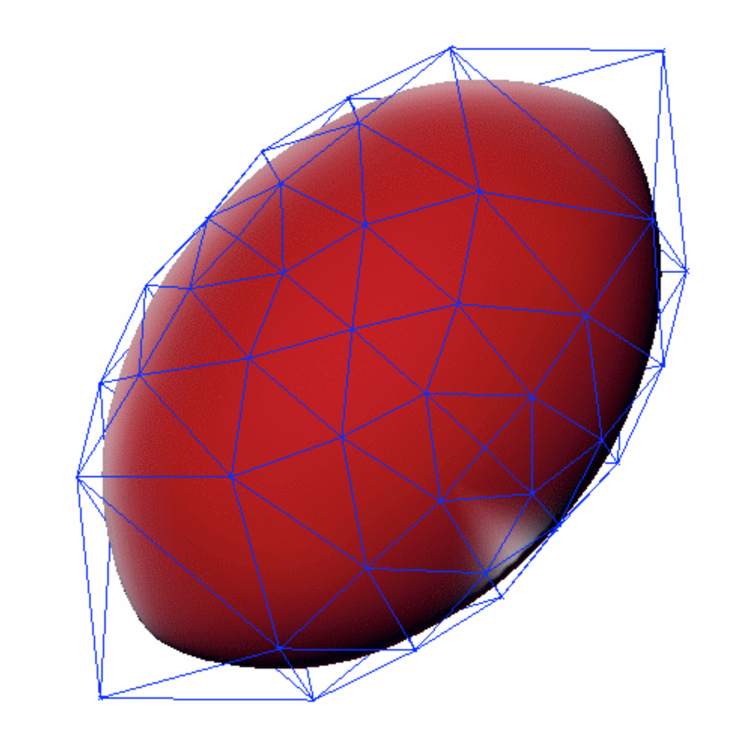




Limit
surface

Change Shape by Diffusion Tensor





$$a(x) = x_1^2 + x_2^2; \quad \text{where} \quad x = (x_1; x_2; x_3)$$

Enhance Sharp Features

Let $v^{(1)}(x), v^{(2)}(x),$ be the principle directions of $M(t)$ at print $x(t)$.

$N(x)$ Be the normal at that point.

Then any vector z in the tangent plane could be expressed as

$$z = \alpha v^{(1)}(x) + \beta v^{(2)}(x) + \delta N(x)$$

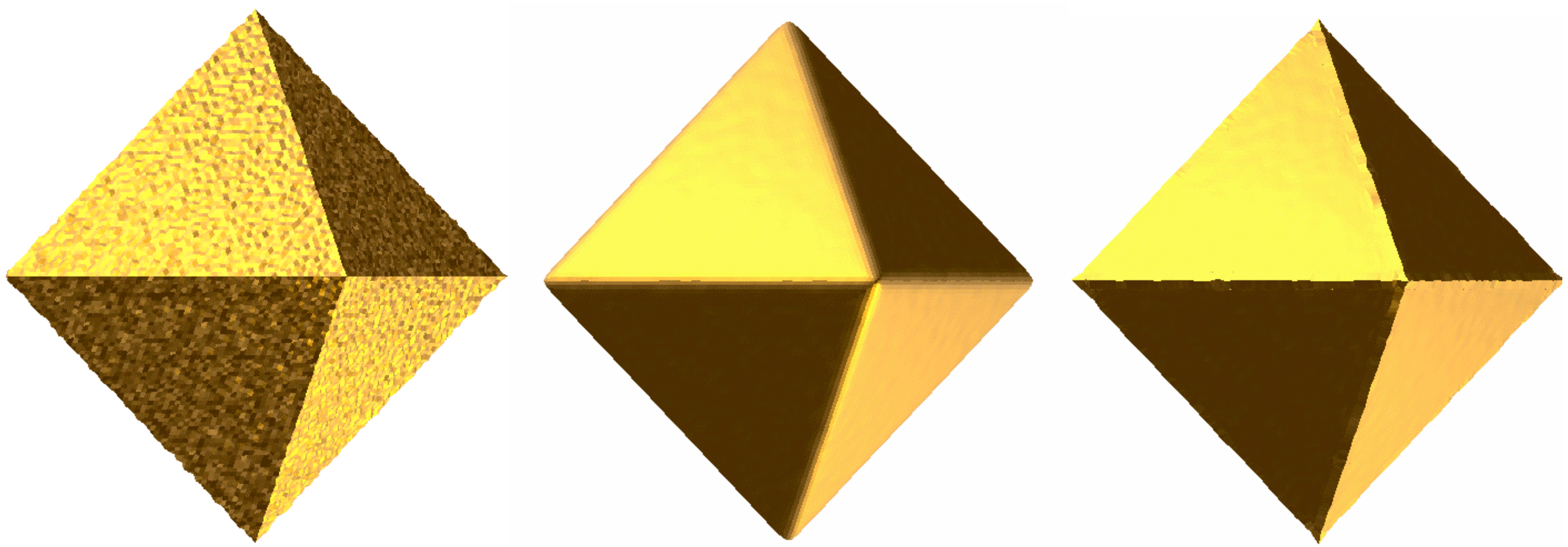
Then define a, such that

$$az = g(k_1)\alpha v^{(1)}(x) + g(k_2)\beta v^{(2)}(x) + \delta N(x)$$

where

$$g(s) = \begin{cases} 1, & s \le \lambda \\ 2(1+\frac{s^2}{\lambda^2})^{-1}, & s > \lambda \end{cases}$$

$\lambda > 0$ is given constant.



Evolution Equation

Let $\Phi_0(x, y)$ be gray-level value, introducing an artificial time *t*, the image deforms according to

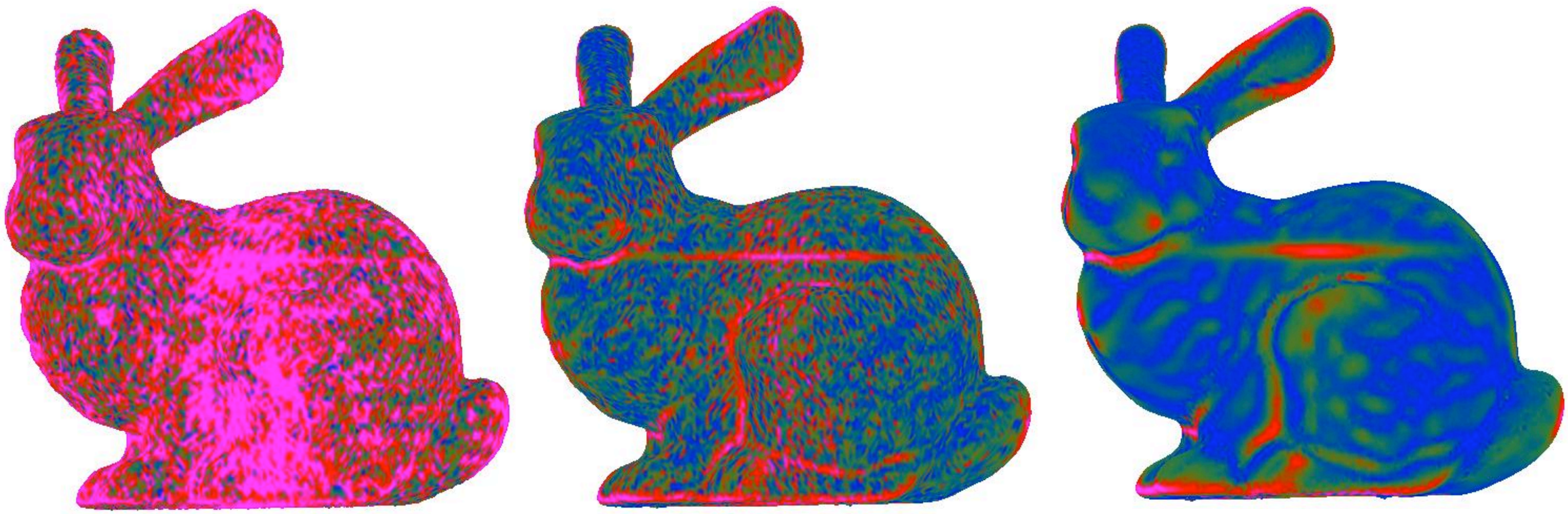
$$\frac{\partial \Phi}{\partial t} = \mathcal{F}[\Phi(x, y, t)],$$

where $\Phi(x, y, t) : R^2 \times [0, \tau) \to R$ is the evolving image,

$\mathcal{F} : R \to R$ is an operator that characterizes the given algorithm,

and the image Φ_0 is the initial condition.

Mean Curvature Plot



Initial data | After 1 iteration | After 4 iterations

Center for Computational Visualization
Institute of Computational and Engineering Sciences
Department of Computer Sciences
University of Texas at Austin

Curvature Driven Evolution

For curves or surfaces

$$\frac{\partial \Phi}{\partial t} = \mathcal{F}(k_i)\mathcal{N}$$

where k_i are the principal curvatures and $\mathcal{N}$ is the normal. This equation describes the deformation of curves or surfaces in its normal direction.

Variational Problem

Assume a variational approach formulated as

$$\arg\{\mathrm{Min}_{\Phi}\mathcal{U}(\Phi)\}$$

where $\mathcal{U}$ is given energy. Let $\mathcal{F}(\Phi)$ denote the Euler-Lagrange derivatives. Since under general assumptions, a necessary condition for Φ to be a minimizier of $\mathcal{U}$ is that $\mathcal{F}(\Phi) = 0$, the minima may be computed via the steady solution of the equation

$$\frac{\partial\Phi}{\partial t} = \mathcal{F}(\Phi)$$

where t is an artificial time parameter.

Algorithms Combination

If two different image processing schemes are given by

$$\frac{\partial \Phi}{\partial t} = \mathcal{F}_1(\Phi), \qquad \frac{\partial \Phi}{\partial t} = \mathcal{F}_2(\Phi)$$

then they can be combined as

$$\frac{\partial \Phi}{\partial t} = \alpha \mathcal{F}_1(\Phi) + \mathcal{F}_2(\Phi)$$

Overview: PDE based diffusion

- Heat equation

$$\partial_t \phi - \mathrm{div} \nabla \phi = 0$$

the solution is

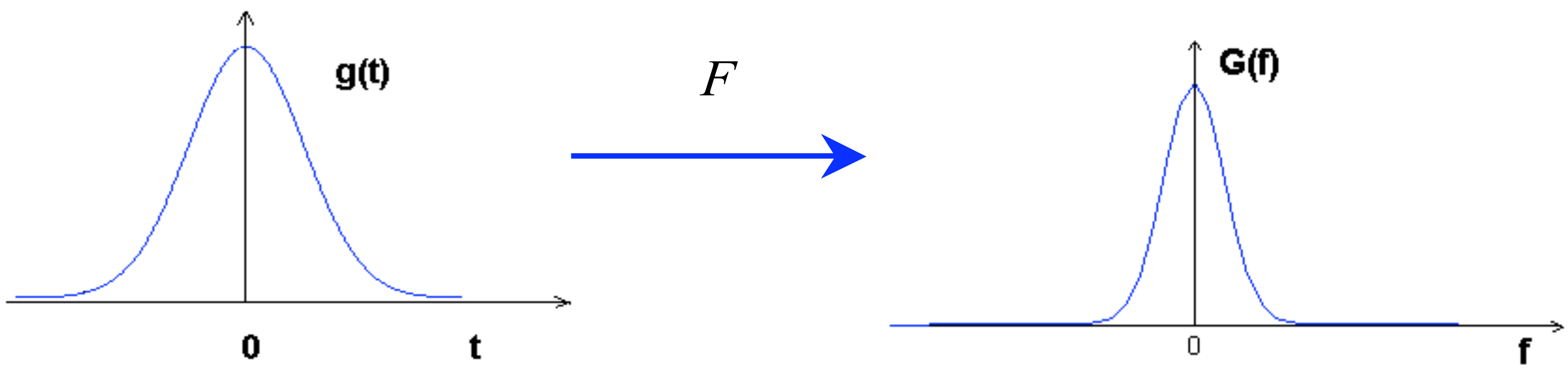
$$\phi(t) = \begin{cases} \phi_0 & (\mathrm{t} = 0) \\ K_{\sqrt{2t}} * \phi_0 & (\mathrm{t} > 0) \end{cases}$$

where $K_\sigma(.)$ denotes the Gaussian filter of width σ

Overview: Gaussian Filter

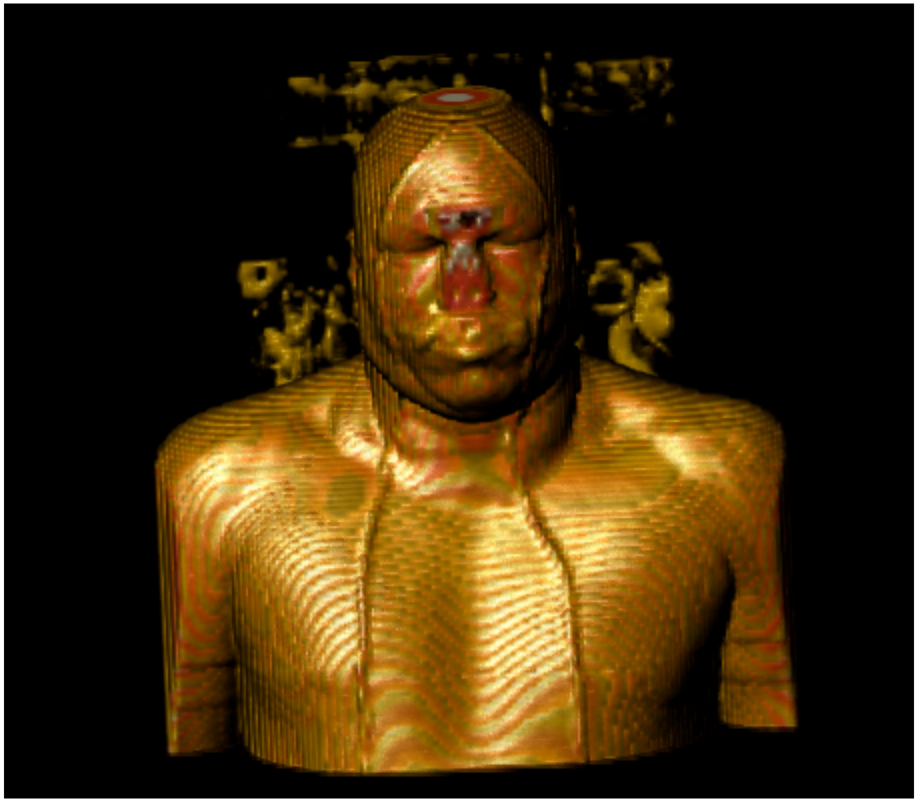
$$g(t) = e^{-\alpha t^2}$$

$$G(f) = \sqrt{\frac{\pi}{\alpha}} e^{-\pi^2 f^2 / \alpha}$$

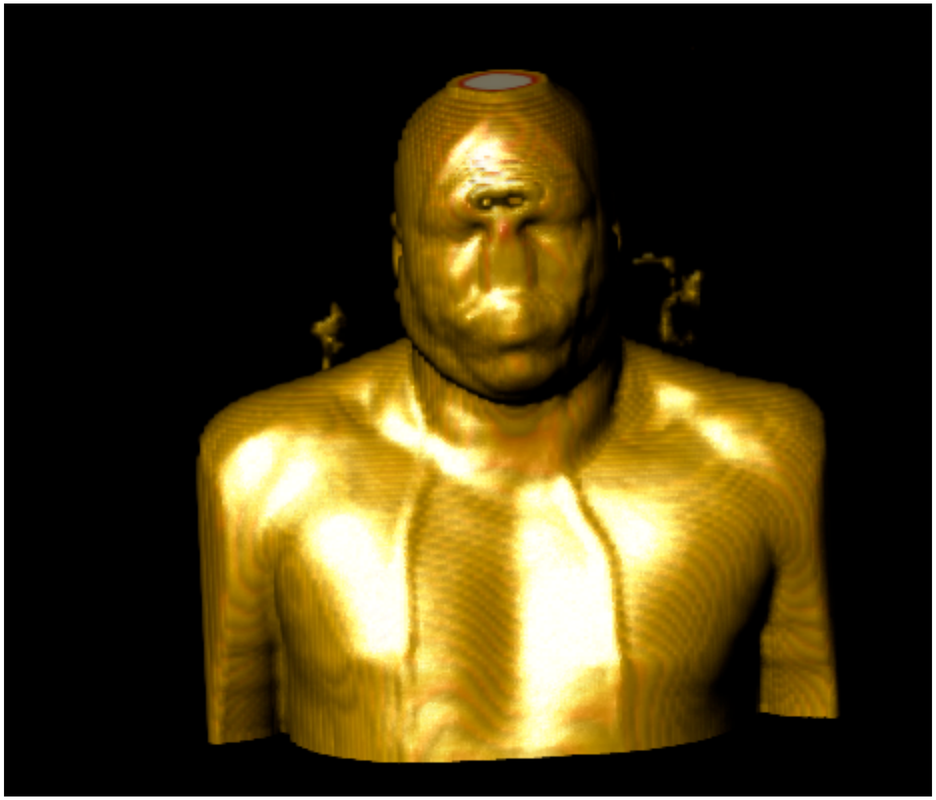


Low Pass Filter

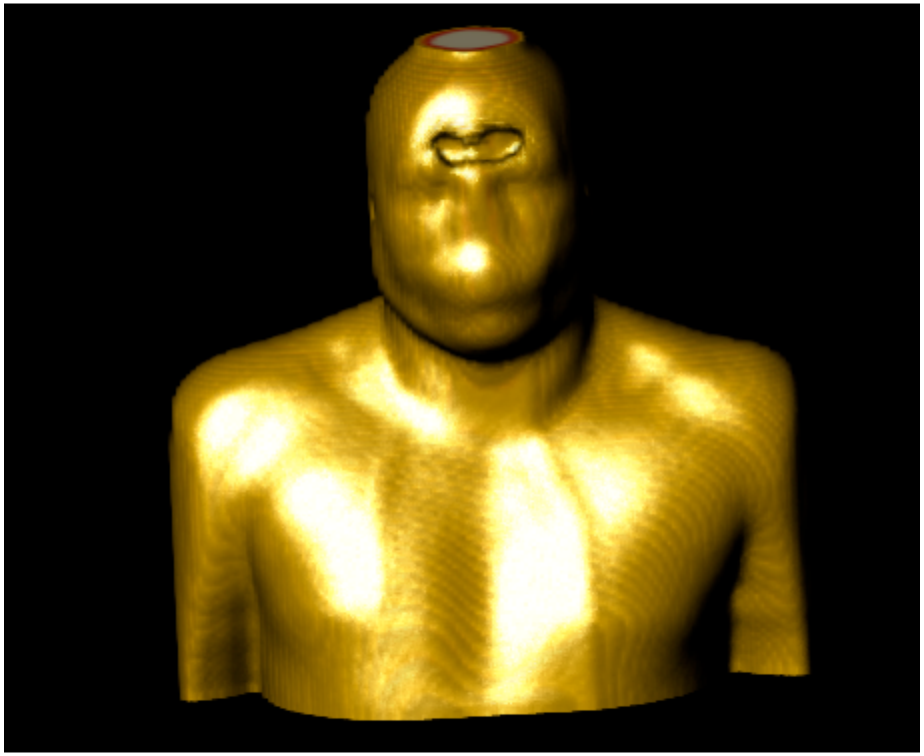
Volumetric Image Rendering



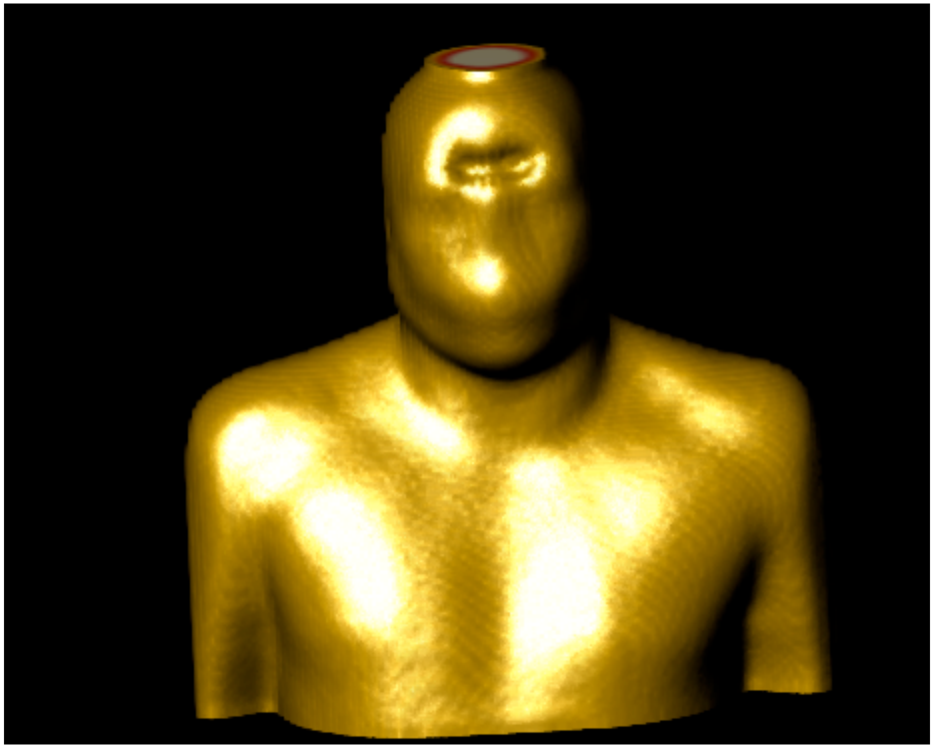
Isotropic Diffusion (1 timestep)



Isotropic Diffusion (3 timesteps)



Isotropic Diffusion (24 timestep)



Nonlinear Image Diffusion

- Early attempt: Perona-Malik model

$$\partial_t \phi - \mathrm{div}(\mathrm{g}(|\nabla \phi|)\nabla \phi) = 0$$

where diffusivity g becomes small for large $|\nabla \phi|$, i.e. at edges

$$g(|\nabla \phi|) = \frac{1}{1 + |\nabla \phi|^2 / \lambda^2}$$

or

$$g(|\nabla \phi|) = \exp(-\frac{|\nabla \phi|^2}{\lambda^2})$$

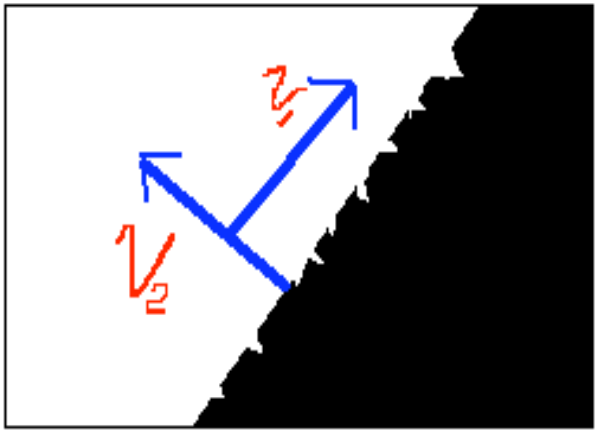
Anisotropic Diffusion

Weickert's anisotropic model:

Local structure: $\nabla\phi_\delta$

- Eigenvectors: $v_1 \parallel \nabla\phi_\delta$ $v_2 \perp \nabla\phi_\delta$
- Diffusivity along edges $\lambda_1 = 1$
- Inhibit diffusivity across edges

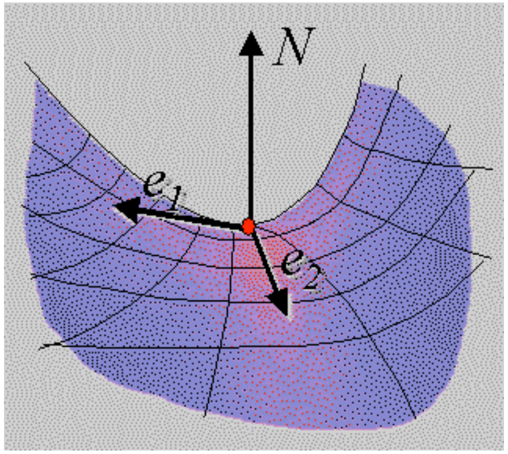
$$\lambda_2 = \frac{1}{1 + |\nabla\phi_\delta|^2 / \lambda^2}$$



Anisotropic Volume Diffusion

Preuβer and Rumpf's level set method for anisotropic geometric diffusion

$$\partial_t \phi - \|\nabla \phi\| div(\mathbf{D}^\sigma \frac{\nabla \phi}{\|\nabla \phi_\sigma\|}) = 0$$



decompose any local vector into three directions:

two principal directions of curvature

normal direction of local structure

Level set based Geometric Diffusion

- Diffusion tensor

$$\mathbf{D}^{\sigma} = \mathbf{B}_{\sigma}^{T} \begin{pmatrix} G_{1,2}(\kappa^{1,\sigma}) & & \\ & G_{1,2}(\kappa^{2,\sigma}) & \\ & & 0 \end{pmatrix} \mathbf{B}_{\sigma}$$

- Curvatures enhancing(1D features) along two principal directions of curvature on surface
- No smoothing along normal direction

Level set based Geometric Diffusion

- Any vector can be decomposed as

$$Z = \alpha v_1 + \beta v_2 + \gamma N$$

- Then

$$\mathbf{D}Z = \alpha g(\kappa_1) v_1 + \beta g(\kappa_2) v_2 + \gamma 0 N$$

- so

$$D\nabla\Phi = <v_1, N> g(\kappa_1) v_1 + <v_2, N> g(\kappa_2) v_2 = \mathbf{0}$$

Anisotropic Volumetric Diffusion: 3D curvature

- Three principal directions of curvature from volumetric image--hypersurface in 4D
- use Gram-Schmidt to construct an orthogonal frame of tangent space (e_1, e_2, e_3)
- the mean curvature vector at point x is

$$H(x) = \frac{1}{3}[h(e_1, e_1) + h(e_2, e_2) + h(e_3, e_3)]$$

- where $$h(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$$

Anisotropic Volume Diffusion: Mean Curvature Vector

∇ and $\tilde{\nabla}$ and are the Riemannian connection in M and R^k respectively

TM is the tangent space $TM^{\perp}$ is the normal space

- Since $\nabla_X Y \in TM$ and $h(X,Y) \in TM^{\perp}$

  only computation of $\tilde{\nabla}_X Y$ is considered and then projected into the normal space to obtain $h(X,Y)$

- Mean curvature vector $H(x) = (\tilde{\nabla}_{e_1} e_1 + \tilde{\nabla}_{e_2} e_2 + \tilde{\nabla}_{e_3} e_3)^{\perp}$

  $\perp$ denotes the normal component of a vector.

The Second Fundamental Form (Tensor)

- Calculate the second fundamental form
- Let n be a normal vector field on M and X be a vector field tangent to M, according to the *equation of Weingarten*, we have

$$\tilde{\nabla}_X n = -A_h X + \nabla^{\perp}_X n$$

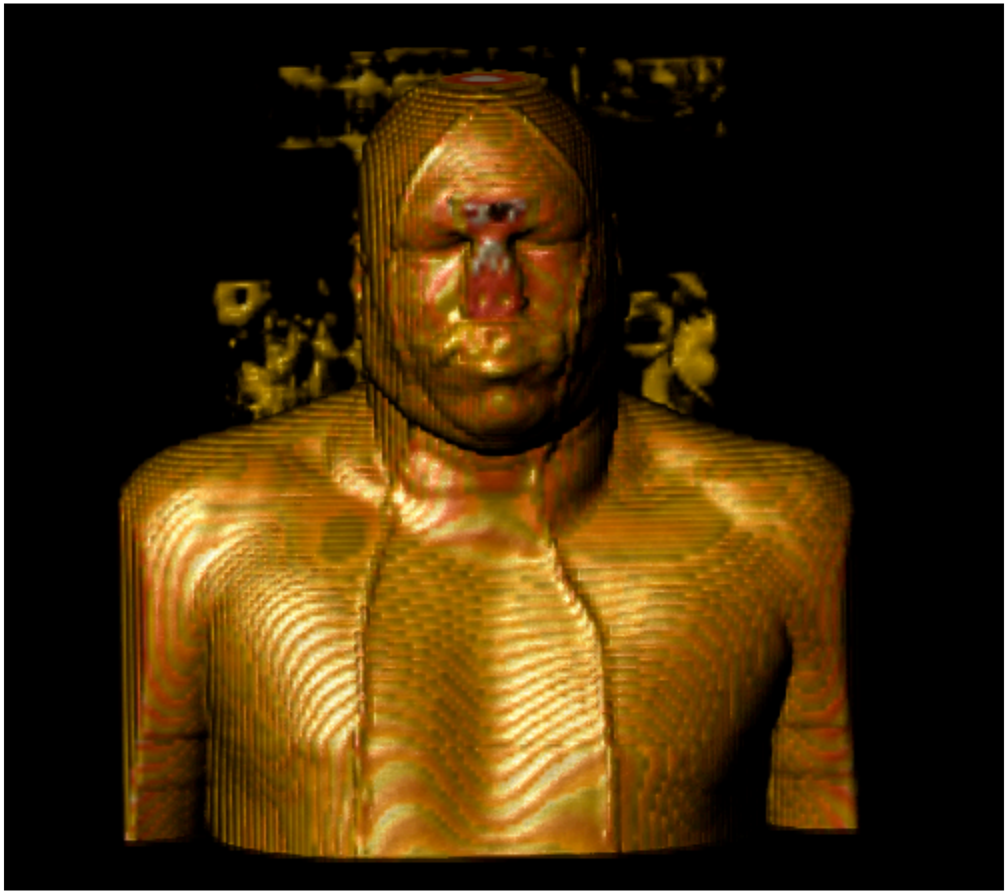
- where $-A_h X$ and $\nabla^{\perp}_X n$ are respectively the tangent and normal components

Principal Curvatures and Directions

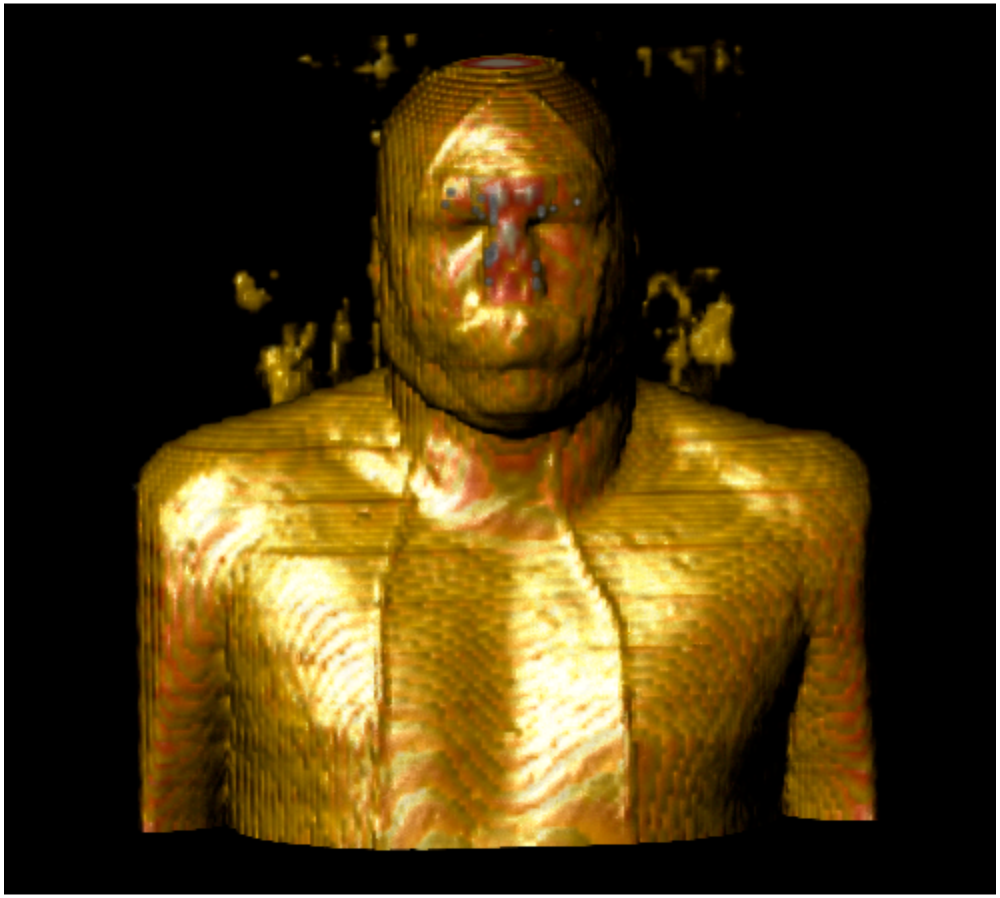
- The principal directions of curvature $\{\upsilon^1, \upsilon^2, \upsilon^3\}$ are the unit eigenvectors of matrix A_h
- Principal curvatures $\{\kappa_1, \kappa_2, \kappa_3\}$ are the corresponding eigenvalues
- Anisotropic diffusion tensor

$$D^{\varepsilon} = [v_1, v_2, v_3]^T \begin{bmatrix} G(\kappa_1) & 0 & 0 \\ 0 & G(\kappa_2) & 0 \\ 0 & 0 & G(\kappa_3) \end{bmatrix} [v_1, v_2, v_3]$$

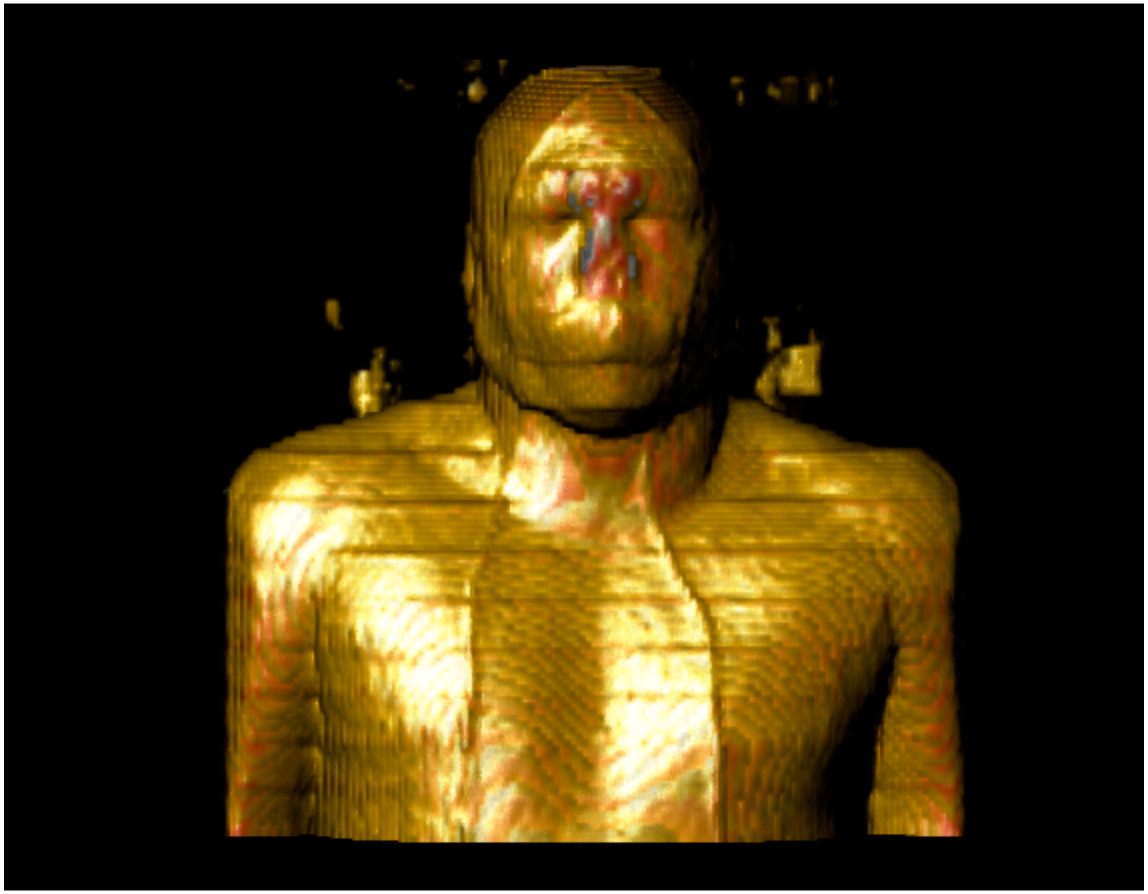
Volumetric Image Rendering (Original Data)



Anisotropic Volume Diffusion (1 timestep)



Anisotropic Volume Diffusion (5 timesteps)



Finite Element Method

- Discretize

$$\sum_p \partial_t X_p(t)\left(\frac{N_p}{\|\nabla\Phi\|}, N_q\right) + \varepsilon \sum_p X_p(t)\left(D^\rho \frac{\nabla N_p}{\|\nabla\Phi\|}, \nabla N_q\right) = 0$$

Result

$$(M^n + \tau L^n(D_\varepsilon^n))X^{n+1} = M^n X^n$$

Where

$$M = \left(\frac{N_p}{\|\nabla\Phi\|}, N_q\right)_{p,q} \qquad L = \frac{(D\nabla N_p, \nabla N_q)_{p,q}}{\|\nabla\Phi^\varepsilon\|}$$

Reading

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   Anisotropic Diffusion of Surfaces and Functions on Surfaces
   ACM Transactions on Graphics, 22, 1, (2003), 4-32.
2. G. Xu, Q. Pan, C. Bajaj **Discrete Surface Modelling Using Partial Differential Equations** *Computer Aided Geometric Design, Volume 23/2, pp 125-145, 2006.*
3. C. Bajaj, J. Chen, R. Holt, A. Netravali **Energy Formulations of A-Splines** *Computer Aided Geometric Design, 16:1(1999), 39-59.*
4. C. Bajaj, G. Xu, Q. Zhang **Bio-Molecule Surfaces Construction Via a Higher-Order Level Set Method** *Proceedings of the 14th CAD/CG International Conference, 2007, Beijing, China*