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Automatic parameterization of rational curves and surfaces III: Algebraic plane curves

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Abstract. We present algorithms to compute the genus and rational parametric equations, for implicitly defined irreducible plane algebraic curves of arbitrary degree. Rational parameterizations exist for all irreducible algebraic curves of genus 0. The genus is computed by a complete analysis of the singularities of plane algebraic curves, using affine quadratic transformations. The rational parameterization techniques, essentially, reduce to solving symbolically systems of homogeneous linear equations and the computation of polynomial resultants.

1. Introduction

Effective computations with algebraic curves and surfaces are increasingly proving useful in the domain of geometric modeling and computer graphics where current research is involved in increasing the geometric coverage of solids to be modeled and displayed, to include algebraic curves and surfaces of arbitrary degree, see [Sederberg '84], [Hoffmann & Hopcroft '85], [de Montaudoin, Tiller & Vold '86], [Farouki '86], [Bajaj '88]. An irreducible algebraic plane curve is implicitly defined by a single prime polynomial equation $f(x, y) = 0$ over an algebraically closed field of characteristic zero, such as the field of complex numbers. Certain plane algebraic curves have an alternate representation, namely the rational parametric equations which are given as $(x(t), y(t))$, where $x(t)$ and $y(t)$ are rational functions in t, i.e., the quotient of polynomials in t.

Both implicit and parametric representations have their inherent advantages. It is thus important to design algorithms for both these curve representations as well as algorithms to convert efficiently from one to the other, whenever possible. Though all algebraic curves have an implicit representation, only irreducible algebraic curves with *genus* = 0 are rational, i.e., have a rational parametric representation, (see [Walker '78], p. 188). The genus of the curve measures the deficiency of singularities on the curve from its maximum allowable limit. A method of computing the genus of irreducible plane algebraic curves is presented in this paper, which uses affine quadratic transformations.

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Recently, various efficient methods have been given for obtaining the parametric equations for special low degree irreducible rational algebraic curves: degree two and three plane algebraic curves [Abhyankar & Bajaj '87a, b], the rational space curves arising from the intersection of degree two surfaces [Levin '79], [Ocken, Schwartz & Sharir '86]. The parameterization algorithms presented in this paper are applicable for implicitly defined irreducible rational plane algebraic curves of arbitrary degree. The computed rational parameterization is over the traditional power basis; however, one may convert this to an equivalent Bernstein form over an arbitrary parameter range, by using the conversion algorithms of [Geisow '83].

The reverse problem of converting from parametric to implicit equations for plane algebraic curves, called implicitization, is achieved by eliminating the single parameter from the two parametric equations, see [Rowe '17], [Sederberg, Anderson & Goldman '84], [Bajaj '87]. This eliminant is also known as the Sylvester resultant, see [Salmon 1885]. Efficient computation of the Sylvester resultant, has been considered by various authors: for univariate polynomials see [Schwartz '80], [Brent, Gustavson & Yun '80] and for multivariate polynomials see [Collins '71], [Bajaj & Royappa '88].

The rest of this paper is as follows. In Section 2 we examine the intricate relationship of *genus* with curve rational parameterizations and also describe a method of computing the genus of irreducible algebraic plane curves. Examples of rational curves are: conics (degree 2 curves); cubics with a singular (double) point; quartics with three double point singularities, etc. In Section 3 we present an efficient algorithm to construct rational parameterizations for a special class of plane curves. These parameterizations are obtained by taking lines through a singular point on the curves, with the slope of the lines being the parameter. This technique suffices for the rational parameterization of conics, cubics with one double point and all irreducible higher degree d curves with a $(d-1)$-fold singularity. In Section 4 we generalize the algorithm of Section 3 to provide rational parameterizations for all irreducible rational plane curves. These rational parameterizations are obtained by taking a one parameter family (pencil) of curves of degree $d-2$ through fixed points on the original curve of degree d. Crucial here is the distinction between *distinct* and *infinitely near* singularities of an algebraic plane curve. Various algorithmic algebraic techniques are illustrated, such as the mapping of points to infinity, the 'passing' of a pencil of curves through fixed points, the 'blowing up' of singularities by affine quadratic transformations, etc.

2. Genus and parameterization

An irreducible algebraic curve C_d of degree d in the plane is one which intersects most lines in d points. Lines through a point P intersect C_d (outside P) in general at $d - \text{mult}_p C_d$ points, where $\text{mult}_p C_d$ = multiplicity of C_d at $P = e$ = order at P of the polynomial equation describing C_d. The order of a polynomial equation at a point P with coordinates (a, b), is the minimum $(i+j)$, when the polynomial is expressed in terms of $(x-a)^i(y-b)^j$. If $e=1$, then P is called a simple point. If $e>1$, then P is a *singular point* of the curve C_d with multiplicity e. If $e=2$, then P is also called a double point. In general, we talk about an e-ple point or an e-fold point. This then leads to the following well known theorem for curves

Theorem 1 (Bezout). *A curve of degree d_1 and a curve of degree d_2, with no common components, meet at $d_1 d_2$ points, counting multiplicities and points at infinity.* ($C_{d_1} \cdot C_{d_2} = d_1 d_2$ *points.*)

Consider a curve C_d described by a polynomial equation $f(x, y) = 0$ of degree d and with order e at the origin.

$$C_d:\quad f(x, y) = \sum_{e \le i+j \le d} a_{ij} x^i y^j = f_d(x, y) + f_{d-1}(x, y) + \cdots + f_e(x, y)$$

(where $f_i(x, y)$ are homogeneous polynomials of degree i together with $f_d(x, y) \neq 0$ and $f_e(x, y) \neq 0$, so that $d=$ degree and $e=$ order). Here $f_d(x, y)$ is also called the degree form and $f_e(x, y)$ is called the initial or order form. The equation of a line through the origin is $y = tx$. Its intersection with the curve is given by

$$\begin{aligned} f(x, tx) &= f_d(x, tx) + f_{d-1}(x, tx) + \cdots + f_e(x, tx) \\ &= x^d f_d(1, t) + x^{d-1} f_{d-1}(1, t) + \cdots + x^e f_e(1, t) \\ &= x^e\left[f_d(1, t) x^{d-e} + \cdots + f_e(1, t)\right]. \end{aligned}$$

Lines through the origin intersect the curve, outside the origin, in $d-e$ points. Hence the multiplicity of the origin $= e$ ($=$ *order* of the polynomial equation describing the curve). Note that by translation, {if (a, b) is the point P, then by setting $\bar{x} = x - a$, $\bar{y} = y - b$} we can assume any point P on the curve to be the origin. Thus if the curve C_d has a $(d-1)$-fold point at the origin $(e = d-1)$, then lines $y = tx$ through the origin intersect f at one other point, and hence x and y can be expressed as rational functions of one parameter t, i.e., rationally parameterizes the curve.

Here we can also note that for most values of m, $f_e(1, m) \neq 0$. The values of m for which it is zero correspond to the tangents to the curve at the origin, $[f_e(x, y) = \prod_{i=1}^{e}(y - m_i x)]$. (Tangentus at P are thus those special lines which intersect C_d at P at more than e points, where $e =$ multiplicity of C_d at P.)

Now note, for example, that the equation of a conic has five independent coefficients and if we take five 'independent' points in the plane and consider a conic passing through these points then this will give five linear homogeneous equations in the five coefficient variables. If the rank of the system matrix is 5, then there is a unique conic through these points. In general, the number of independent coefficients of a plane algebraic curve C_d of degree d is $\frac{1}{2}d(d+3)$.

Using conics one can for example, easily prove by Bezout's theorem that a curve of degree 4 cannot have 4 double points. In general one may see that the number of double points, say DP, of C_d is $\leq \frac{1}{2}(d-1)(d-2)$. Assume $DP > \frac{1}{2}(d-1)(d-2)$. Then since $\frac{1}{2}(d-2)(d+1)$ fixed points determine a C_{d-2} curve and if we choose $\frac{1}{2}(d-1)(d-2)+1$ double points of C_d, then to determine C_{d-2} one needs a remaining

$$\tfrac{1}{2}(d-2)(d+1) - \left(\tfrac{1}{2}(d-1)(d-2)+1\right) = (d-2) - 1 = d-3 \text{ points.}$$

So take $d-3$ other fixed simple points of C_d. Then we can pass a C_{d-2} curve through the above $\frac{1}{2}(d-1)(d-2)+1$ double points of C_d and $d-3$ other simple points of C_d. Then counting the number of points of intersection of C_d and C_{d-2}, together with multiplicities, yields

$$(d-1)(d-2) + 2 + d - 3 = d^2 - 2d + 1 = (d-2)d + 1 = C_d \cdot C_{d-2} + 1$$

which contradicts Bezout. Thus assuming Bezout we see that

$$DP \leq \tfrac{1}{2}(d-1)(d-2).$$

In general, we have Table 1.

One definition of the *genus* g of a curve C_d is a measure of how much the curve is deficient from its maximum allowable limit of singularities,

$$g = \tfrac{1}{2}(d-1)(d-2) - DP$$

where DP is a 'proper' counting of the number of double points of C_d (summing over all singularities). From the earlier discussion and Bezout, we can see that in counting the number of double points DP of C_d an e-ple point of C is to be counted as $\frac{1}{2}e(e-1)$ double points.

However this counting is not very precise as such is the case only for the so called *distinct* multiple points of C. For a multiple point, that is not *distinct*, one has also to consider *infinitely*

Table 1

degree of curve	1	2	3	4	5	6	...	d
the maximum number of double points	0	0	1	3	6	10	...	$\frac{1}{2}(d-1)(d-2)$
the number of independent paramaters	2	5	9	14	20	27	...	$\frac{1}{2}d(d+3)$

near singularities. In general a double point is roughly either a *node* or a *cusp*. If a cusp is given by $y^2 - x^3$ it is called a *distinct* cusp and is counted as a single double point. Cusps other than distinct look like $y^2 - x^{2m+1}$ (an *m-fold* cusp). Though the multiplicity of the origin is two the origin accounts for m double points when counted properly. The proper counting was achieved by Noether using homogeneous 'Cremona quadratic transformations', see also [Walker '78]. Following [Abhyankar '83] we can achieve the same thing by using 'affine quadratic transformations'.

Consider for example, the cusp $y^2 - x^3 = 0$ which has a double point at the origin. The quadratic transformation [1] (or substitution) $\bar{q}$ given by

$$x = \bar{x} \quad \text{and} \quad y = \bar{x}\,\bar{y} \tag{1}$$

yields

$$0 = y^2 - x^3 = \bar{x}^2\bar{y}^2 - \bar{x}^3 = \bar{x}^2(\bar{y}^2 - \bar{x}),$$

and cancelling out the extraneous factor $\bar{x}^2$ we get the nonsingular parabola $\bar{y}^2 - \bar{x} = 0$. So the origin in this case was a *distinct* singular point and is counted as a single double point. To desingularize the *m-fold* cusp one has to make a succession of m transformations of the type (1). Only the mth successive application of (1) changes the multiplicity of the origin from two to one. Hence in this case, counting properly, we say that the cusp has one *distinct* double point and $(m-1)$ *infinitely near* double points, giving a total *DP* count of m.

In a general procedure for counting double points, given an e-fold point P of a plane curve C, we choose our coordinates to bring P to the origin and then apply (1). If now C: $f(x, y) = 0$, then the substitution (1) transforms C into the curve $\bar{C}$: $\bar{f}(\bar{x}, \bar{y}) = 0$ given by

$$f(\bar{x}, \bar{x}\,\bar{y}) = \bar{x}^e \bar{f}(\bar{x}, \bar{y}).$$

$\bar{C}$ will meet the line E: $\bar{x} = 0$ in the points $P^1, \dots, P^m$, the roots of $\bar{f}(0, y) = 0$ which corresponds to the tangents to C at P. If P^i is an e_i-fold point of $\bar{C}$, then we shall have $e_1 + \cdots + e_m \leq e$. We say that $P^1, \dots, P^m$ are the points of $\bar{C}$ in the first neighborhood of P, and the multiplicity of $\bar{C}$ at P^i is e_i. Now iterate this procedure. The points of C *infinitely near* P can be diagrammed by the *singularity tree* of C at P (see Fig. 1).

At every node of this tree (including the root) we keep a count equal to the multiplicity of C at that point, which will then be greater than or equal to the number of branches arising at that node. It follows that every node higher than a certain level will be unforked, that is have a single branch. The desingularization theorem for algebraic plane curves, see [Abhyankar '83], or [Walker '78], says that at every node higher than a certain level, the count equals one. In other words, C has only a finite number of singularities infinitely near P. Thus, since C has only

[1] The quadratic transformation $\bar{q}$ maps the origin to the line $\bar{x} = 0$, and is one-to-one for all points (x, y) with $x \neq 0$. Viewed alternatively, $\bar{q}$ maps tangent directions to f at the origin to different points on the exceptional line $\bar{x} = 0$. This may be seen by noting that the lines $y = mx$ are mapped to parallel lines $\bar{y} = m$ which intersect the exceptional line at points $(0, m)$. But $\bar{q}$ does not map the line $x = 0$ properly, so we must make sure that $x = 0$ is not a tangent direction to the curve at the origin. This is done by a nonsingular transformation $x = u\hat{x} + v\hat{y}$ and $y = r\hat{x} + s\hat{y}$ where neither $u\hat{x} + v\hat{y}$ nor $r\hat{x} + s\hat{y}$ are tangents to f at the origin.

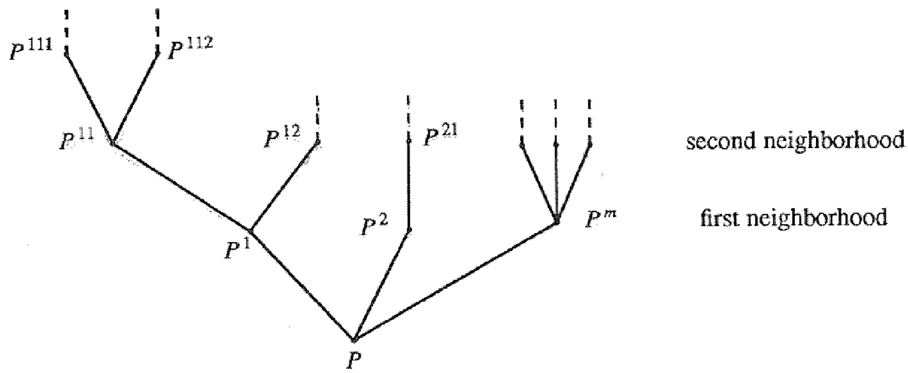

Fig. 1. Singularity tree.

finitely many *distinct* singularities, it follows that C has only a finite number of singular points, *distinct* as well as *infinitely near*.

Thus, by summing the counts of each node and counting $\frac{1}{2}e(e-1)$ double points for a count e and additionally summing over all singularities of C and their corresponding singularity trees, we obtain a precise count of the total number of double points DP of C. This proper counting of double points then yields the following theorem

Theorem 2 (Cayley–Riemann). $g = 0$ *iff* C *has a rational parameterization.*

In other words the given plane curve has its maximum allowable limit of singularities, if and only if it is rational.

Note also that in counting singularities we consider all the singularities of the complex projective curve. That is we consider the real and complex singularities at both finite distance as well as at infinity. The process of considering singularities at infinity is no different than that at finite distance. With homogeneous coordinates (X, Y, Z) corresponding to the affine coordinates (x, y), and with $x = X/Z$ and $y = Y/Z$, let us consider $Z = 0$ to be the line at infinity. By swapping one of the axis lines $X = 0$ or $Y = 0$ with the line at infinity we can bring the points at infinity to the affine plane. We illustrate this as well as *Theorem 2* by means of an example. Consider again the *m-fold* cusp $y^2 - x^{2m+1}$. We have seen earlier that the origin accounts for m double points when counted properly. Now consider the singularity at infinity. We swap the $Z = 0$ line with the $Y = 0$ line by homogenizing and then setting $Y = 1$ to dehomogenize:

$$Y^2Z^{2m-1} - X^{2m+1} \quad \Rightarrow \quad z^{2m-1} - x^{2m+1}.$$

The singularity at infinity is again at the origin and of multiplicity $2m - 1$ accounting for $\frac{1}{2}(2m-1)(2m-2)$ double points. On applying an appropriate quadratic transformation $x = \bar{x}$ and $z = \bar{x}\bar{z}$, the multiplicity at the origin is reduced to 2:

$$\bar{z}^{2m-1} - \bar{x}^2.$$

After a sequence of $m - 1$ additional quadratic transformations the multiplicity at the original finally reduces to one. These *infinitely near* singularities then account for totally $m - 1$ additional double points, resulting in a total DP count for the curve to be equal to

$$m + \tfrac{1}{2}(2m-1)(2m-2) + m - 1 = \tfrac{1}{2}(2m)(2m-1)$$

which is exactly the maximum number of allowable double points for a curve of degree $2m + 1$. Hence the m-fold cusp has genus 0 and is rational with a parameterization given by

$$x = t^2, \qquad y = t^{2m+1}.$$

3. Parameterizing with lines

The geometric idea of parametrizing a circle or a conic is to fix a point and take a pencil of lines through that point which meet the conic at one additional point. Hence conics always have a rational parameterization, with the slope of the line being the single parameter. Next, consider a cubic curve, C_3 which is a curve with which most lines intersect in three points. If we consider a *singular* cubic curve (genus 0), SC_3, then a pencil of lines through the singular (double) point intersects SC_3 at one additional point and hence rationally parameterizes SC_3. If C_3 has no singular points (genus 1), then C_3 cannot be parameterized by rational functions.

Now intersecting a curve C with a pencil of lines through a fixed point P on it, can efficiently be achieved by sending the point P on C to infinity. To understand this, let us first consider an irreducible conic which is represented by the equation

$$g(x, y) = ax^2 + by^2 + cxy + dx + ey + f = 0.$$

Bezout confirms that the irreducible conic cannot contain a double point for otherwise the conic consists of two lines. We observe that the trivial parameterizable cases are the parabola $y^2 = x$ which has no term in x^2; the parabola $x^2 = y$ which has no term in y^2; and the hyperbola $xy = 1$ which has no terms in x^2 and y^2. The non-trivial case arises when a and b are both non-zero, e.g. the ellipse. This then suggests that to obtain a rational parameterization all we need to do is to kill the y^2 term. This can always be achieved by a suitable linear coordinate transformation resulting in the equation

$$(rx + s)y + (ux^2 + vx + w) = 0.$$

from which one could easily obtain a rational parametrization

$$x = t, \qquad y = \frac{-(ut^2 + vt + w)}{(rt + s)}.$$

The elimination of the x^2 or the y^2 term through a linear coordinate transformation is said to make the conic *irregular* in x or y respectively. Geometrically speaking, a conic being irregular in x or y means that most lines parallel to the x or y axis respectively, intersect the conic in one point. Remember that most lines through a fixed point on the conic meet the conic in one additional varying point. By sending the fixed point to infinity we make all these lines parallel to some axis and the curve irregular in one of the variables (x or y) and hence amenable to parameterization. The coordinate transformation we select is thus one which sends any point on the conic to infinity along either of the coordinate axes x or y.

As an example consider the unit circle and fix a simple point P $(-1, 0)$ on it:

x, y affine coordinates $(-1, 0)$,

X, Y, Z homogeneous coordinates $(-1, 0, 1)$,

and send P to a point at infinity along the y-axis. That is, send $(-1, 0, 1)$ to $(0, 1, 0)$. (Explanation: A point on the y-axis is like $(0, p, 1)$; divide by p giving $(0/p; p/p, 1/p)$ and then let $p \to \infty$ and thereby $(0, 1, 0)$. This we achieve by a homogeneous linear transformation which transforms $(-1, 0, 1)$ to $(0, 1, 0)$

$$X \to \alpha\overline{X} + \beta\overline{Y} + \gamma\overline{Z},$$

$$Y \to \hat{\alpha}\overline{X} + \hat{\beta}\overline{Y} + \hat{\gamma}\overline{Z},$$

$$Z \to \alpha^*\overline{X} + \beta^*\overline{Y} + \gamma^*\overline{Z}.$$

The chosen point on the circle $(-1, 0, 1)$ determines

$$-1 = \beta, \qquad 0 = \hat{\beta}, \qquad 1 = \beta^*$$

and the α's and γ's are chosen such that the det(α's, β's, γ's) $\neq 0$. This yields a well defined invertible linear transformation. So let us take as our homogeneous linear transformation

$$X \to -\overline{Y},$$
$$Y \to \overline{Z},$$
$$Z \to \overline{X} + \overline{Y}.$$

We first homogenize the circle $x^2 + y^2 - 1 = 0$ to $X^2 + Y^2 - Z^2 = 0$. On applying the above linear transformation, we eliminate the $\overline{Y}^2$ term:

$$\overline{Y}^2 + \overline{Z}^2 - (\overline{X} + \overline{Y})^2 = 0$$

$$\Rightarrow \quad -2\overline{X}\overline{Y} = \overline{X}^2 - \overline{Z}^2 \quad \Rightarrow \quad \overline{Y} = \frac{\overline{Z}^2 - \overline{X}^2}{2\overline{X}}.$$

Then on dehomogenizing by setting $\overline{Z} = 1$, using the above linear transformation to obtain the original affine coordinates and setting $\overline{X} = t$ we obtain the rational parameterization of the circle:

$$x = \frac{X}{Z} = \frac{-\overline{Y}}{\overline{X} + \overline{Y}}, \quad y = \frac{Y}{Z} = \frac{1}{\overline{X} + \overline{Y}},$$

$$\begin{cases} \overline{X} = t \\ \overline{Y} = \dfrac{1 - t^2}{2t} \end{cases} \Rightarrow \quad \begin{aligned} x &= \frac{-(1 - t^2)/2t}{t + (1 - t^2)/2t} = -\frac{1 - t^2}{1 + t^2}, \\ y &= \frac{1}{t + (1 - t^2)/2t} = \frac{2t}{1 + t^2}. \end{aligned}$$

In general, curves of degree d with a *distinct* $(d-1)$-fold point, of which the above was a special case, can be rationally parameterized by sending the $(d-1)$-fold point to infinity. Consider $f(x, y)$ a polynomial of degree d in x and y representing a plane algebraic curve C_d of degree d with a *distinct* $(d-1)$-fold singularity.

Note that singularities of a plane curve can be computed by simultaneously solving the equations $f = f_x = f_y = 0$ where f_x and f_y are the x and y partial derivatives of f, respectively. One way of obtaining the common solutions is to find those roots of $\mathrm{Res}_x(f_x, f_y) = 0$ and $\mathrm{Res}_y(f_x, f_y) = 0$ which are also the roots of $f = 0$. Here, $\mathrm{Res}_x(f_x, f_y)$ is the Sylvester resultant of f_x and f_y treating them as polynomials in x, for details see [Bajaj & Royappa '87]. Note singularities at infinity can be obtained, similarly, after replacing the line at infinity with one of the coordinate axes. In particular on homogenizing a plane curve $f(x, y)$ to $F(X, Y, Z)$ we can set $Y = 1$ to obtain $\bar{f}(x, z)$ thereby swapping the line at infinity $Z = 0$ with the line $Y = 0$. Then the above procedure can be applied to $\bar{f}(x, z)$ to find the singularities at infinity.

Let us then compute the $(d-1)$-fold singularity of the given curve C_d and translate it to the origin by a simple linear transformation. Then the polynomial describing the curve will be of the form

$$f(x, y) = f_d(x, y) + f_{d-1}(x, y)$$

where f_d (degree form) consists of the terms of degree d and f_{d-1} consists of terms of degree $d-1$. On homogenizing this curve we obtain

$$F(X, Y, Z) = a_0 Y^d + a_1 Y^{d-1} X + \cdots + a_d X^d$$
$$+ b_0 Y^{d-1} Z + b_1 Y^{d-2} XZ + \cdots + b_d X^{d-1} Z.$$

Now by sending the singular point (0, 0, 1) to infinity along the Y axis we can eliminate the Y^d

term. This is done by a homogeneous linear transformation which maps the point (0, 0, 1) to the point (0, 1, 0) and is given by

$$X = \overline{X}, \qquad Y = \overline{Z}, \qquad Z = \overline{Y}$$

which yields

$$F(\overline{X}, \overline{Y}, \overline{Z}) = a_0\overline{Z}^d + a_1\overline{Z}^{d-1}\overline{X} + \cdots + a_d\overline{X}^d + b_0\overline{Z}^{d-1}\overline{Y} + b_1\overline{Z}^{d-2}\overline{X}\overline{Y} + \cdots + b_d\overline{X}^{d-1}\overline{Y},$$

$$\overline{Y} = -\frac{a_0\overline{Z}^d + a_1\overline{Z}^{d-1}\overline{X} + \cdots + a_d\overline{X}^d}{b_0\overline{Z}^{d-1} + b_1\overline{Z}^{d-2}\overline{X} + \cdots + b_d\overline{X}^{d-1}}.$$

Then dehomogenizing, by setting $\overline{Z} = 1$, and using the linear transformation to obtain the original affine coordinates

$$x = \frac{X}{Z} = \frac{\overline{X}}{\overline{Y}}, \qquad y = \frac{Y}{Z} = \frac{\overline{Z}}{\overline{Y}}$$

and setting $\overline{X} = t$, we obtain the rational parametrization of the curve.

Alternatively we could have symbolically intersected a single parameter family (*pencil*) of lines through the $(d-1)$-fold singularity with C_d and obtained a rational parameterization with respect to this parameter. This concept of passing a pencil of curves through singularities is generalized in the next section.

4. Parameterizing with higher degree curves

From the genus formula and Bezout's theorem we note that an irreducible rational quartic curve in the plane has either a *distinct* triple point or three *distinct* double points etc.. The rational parameterization of the quartic with a *distinct* triple point is handled by the method of Section 3. To give the geometric idea of the method of parameterizing with a pencil of curves, let us consider an irreducible quartic curve C_4 with three *distinct* double points. From Table 1 of Section 2 we know that 5 points define a conic. With the three double points and any simple point of C_4, a one parameter family (pencil) of conics, $C_2(t)$ can be defined. Now $C_4 \cdot C_2(t) = 8$ points. Since the fixed points (3 double points and a simple point) account for $2 + 2 + 2 + 1 = 7$ points, the remaining point on C_4 is the variable point, giving us a rational parametrization of C_4, in terms of parameter t.

Computationally we proceed with homogeneous coordinates allowing us to simultaneously also deal with points at infinity. Consider the projective quartic C_4: $F(X, Y, Z) = 0$ with three distinct double points. We first obtain, by the method sketched in Section 3, the three double point singularities of the homogeneous polynomial $F(X, Y, Z)$ as well as any simple point on it. Let them be given by (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), (X_3, Y_3, Z_3) and (X_4, Y_4, Z_4) respectively. Consider next the general equation of a projective conic C_2 given by

$$G(X, Y, Z) = aX^2 + bY^2 + cXY + dXZ + eYZ + fZ^2 = 0$$

which has six coefficients, however five independent unknowns as we can always divide out by one of the nonzero coefficients. We now try to determine these unknowns to yield a one parameter family of curves, $C_2(t)$. We pass C_2 simply through the singular double points and the simple point of C_4. (In general we shall pass a curve through an *m-fold* singularity with multiplicity $m-1$). In other words we equate for $i = 1, \ldots, 4$,

$$F(X_i, Y_i, Z_i) = G(X_i, Y_i, Z_i) = 0.$$

This yields a linear system of 4 equations in five unknowns. Set one of the unknowns to be t and solve for the remaining unknowns in terms of t.

Next compute the intersection of C_4 and $C_2(t)$, by computing $\mathrm{Res}_Y(F, G)$ which is a polynomial in X, Z and t. On dehomogenizing this polynomial by setting $Z=1$ (since resultants of homogeneous polynomials are homogeneous) and dividing by the common factors $(x-x_i)^2$ for $i=1, 2, 3$ and $(x-x_4)$ we obtain a polynomial linear in x which yields the rational parameterization. The process when repeated for y by taking the $\mathrm{Res}_X(F, G)$ and dividing by the common factors $(y-y_i)^2$ for $i=1, 2, 3$ and $(y-y_4)$ yields a polynomial in y and t which is linear in y and which then yields the rational parameterization.

As an example, consider a quintic curve with also *infinitely near* singularities. In particular, the homogenized quintic cusp C_5: $F(X, Y, Z) = Y^2Z^3 - X^5$ has a *distinct* double point and an *infinitely near* double point (in the first neighborhood) at (0, 0, 1), and a *distinct* triple point and an *infinitely near* double point at (0, 1, 0). Counting all the double points, properly, we see that C_5 has 6 double points and hence is of genus 0 and rational. To obtain the parameterization we pass a one parameter family of cubics $C_3(t)$ given by

$$G(X, Y, Z) = aX^3 + bY^3 + cX^2Y + dXY^2 + eX^2Z + fY^2Z + gXYZ + hXZ^2 + iYZ^2 + jZ^3$$

through the singularities of C_5. Passing $C_3(t)$ through the *distinct* double point (with multiplicity $2-1=1$) is obtained as before by equating

$$F(0, 0, 1) = G(0, 0, 1) = 0, \tag{1}$$

and the *distinct* triple point, (with multiplicity $3-1=2$) by equating

$$F(0, 1, 0) = G(0, 1, 0) = 0, \tag{2}$$

$$F_X(0, 1, 0) = G_X(0, 1, 0) = 0, \tag{3}$$

$$F_Z(0, 1, 0) = G_Z(0, 1, 0) = 0. \tag{4}$$

These conditions for our example curve C_5 makes $j=0$, $b=0$, $d=0$ and $f=0$ in $C_3(t)$ yielding the curve

$$\overline{G}(X, Y, Z) = aX^3 + cX^2Y + eX^2Z + gXYZ + hXZ^2 + iYZ^2.$$

We now wish to pass $C_3(t)$ through the *infinitely near* double point in the first neighborhood of the singularity at (0, 0, 1) of C_5. To achieve this we apply the quadratic transformation $X=\overline{X}$, $Y=\overline{X}\overline{Y}$, $Z=\overline{Z}$ centered at (0, 0, 1) to both $F(X, Y, Z)$ and $\overline{G}(X, Y, Z)$. The transformed equation $F_T = \overline{Y}^2\overline{Z}^3 - \overline{X}^3$ has a double point at (0, 0, 1) and we pass the curve of the transformed equation $\overline{G}_T = a\overline{X}^2 + c\overline{X}^2\overline{Y} + e\overline{X}\overline{Z} + g\overline{X}\overline{Y}\overline{Z} + h\overline{Z}^2 + i\overline{Y}\overline{Z}^2$ through the double point as before by equating

$$F_T(0, 0, 1) = \overline{G}_T(0, 0, 1) = 0. \tag{5}$$

This condition makes $h=0$ in $C_3(t)$ yielding

$$\hat{G}(X, Y, Z) = aX^3 + cX^2Y + eX^2Z + gXYZ + iYZ^2.$$

Similarly we pass C_3 through the *infinitely near* double point in the first neighborhood of the singularity at (0, 1, 0) of C_5. To achieve this we apply the quadratic transformation $X=\hat{X}$, $Y=\hat{Y}$, $Z=\hat{X}\hat{Z}$ centered at (0, 1, 0) to both $F(X, Y, Z)$ and $\hat{G}(X, Y, Z)$. The transformed equation $F_T = \hat{Y}^2\hat{Z}^3 - \hat{X}^2$ has a double point at (0, 1, 0) and we pass the curve of the transformed equation $\hat{G}_T = a\hat{X} + c\hat{Y} + e\hat{X}\hat{Z} + g\hat{Y}\hat{Z} + i\hat{Y}\hat{Z}^2$ through the double point as before by equating

$$F_T(0, 1, 0) = \hat{G}_T(0, 1, 0) = 0. \tag{6}$$

This condition makes $c=0$ in C_3 yielding

$$\tilde{G}(X, Y, Z) = aX^3 + eX^2Z + gXYZ + iYZ^2.$$

Our final condition to determine a pencil of cubics $C_3(t)$ is to choose two simple points on C_5, say (1, 1, 1) and (1, −1, 1) and pass C_3 through then by equating

$$F(1, 1, 1) = \tilde{G}(1, 1, 1) = 0, \tag{7}$$

$$F(1, -1, 1) = \tilde{G}(1, -1, 1) = 0. \tag{8}$$

Note that in total we applied eight conditions to determine the pencil, since nine conditions completely determine the cubic. The last two conditions yield the equations

$$a + e + g + i = 0,$$

$$a + e - g - i = 0.$$

In choosing the pencil $C_3(t)$ we allow one of the coefficients to be t and we may divide out by another coefficient (or choose it to be 1). The above equations yield $a+e=0$ and $g+i=0$ and on choosing $a=t$ and $g=1$ we obtain $e=-t$ and $i=-1$. Hence our homogeneous cubic pencil is given by

$$G_3(X, Y, Z, t) = tX^3 - tX^2Z + XYZ - YZ^2$$

or by the dehomogenized pencil $G_3(x, y, t) = tx^3 - tx^2 + xy - y = 0$. This yields $y = -tx^2$. Intersecting it with the dehomogenized quintic C_5: $y^2 - x^5$ yields $t^2x^4 - x^5 = 0$ or $x = t^2$ on dividing out by the common factor x^4. Finally the parametric equations of the rational quintic C_5 are given by $x = t^2$ and $y = -t^5$.

In the general case we consider an irreducible curve C_d with the appropriate number of *distinct* and *infinitely near* singular points and $d-3$ additional simple points of C_d. Consider again $F(X, Y, Z)$ and $G(X, Y, Z)$ as the homogeneous equations of curves C_d and C_{d-2} respectively. For a distinct singular point of multiplicity m of C_d at the point (X_i, Y_i, Z_i) we pass the curve C_{d-2} through it with a multiplicity of $m-1$. To achieve this we equate

$$F(X_i, Y_i, Z_i) = G(X_i, Y_i, Z_i),$$

$$F_X(X_i, Y_i, Z_i) = G_X(X_i, Y_i, Z_i),$$

$$F_Y(X_i, Y_i, Z_i) = G_Y(X_i, Y_i, Z_i),$$

$$F_{XX}(X_i, Y_i, Z_i) = G_{XX}(X_i, Y_i, Z_i),$$

$$F_{XY}(X_i, Y_i, Z_i) = G_{XY}(X_i, Y_i, Z_i),$$

$$F_{YY}(X_i, Y_i, Z_i) = G_{YY}(X_i, Y_i, Z_i),$$

$$\vdots$$

$$F_{X^jY^k}(X_i, Y_i, Z_i) = G_{X^jY^k}(X_i, Y_i, Z_i), \quad 0 \le j+k \le m-2.$$

For an *infinitely near* singular point of C_d with its associated singularity tree we pass the curve C_{d-2} with multiplicity $r-1$ through each point of the points of multiplicity r in the first, second, third, ..., neighborhoods. To achieve this we apply quadratic transformations T_i to both $F(X, Y, Z)$ and $G(X, Y, Z)$ centered around the *infinitely near* singular points corresponding to the singularity tree. The appropriate multiplicity of passing is achieved by equating the transformed equations F_{T_i} and G_{T_i} and their partial derivatives as above.

A simple counting argument now shows us that this method generates the correct number of conditions which specifies C_{d-2} and furthermore the total intersection count between C_d and C_{d-2} satisfies *Bezout*. A curve C_d of *genus* = 0 has the equivalent of exactly $\frac{1}{2}(d-1)(d-2)$ double points. Then to pass a curve C_{d-2} through these double points and $d-3$ other fixed simple points of C_d and one variable point specified by t, the total number of conditions (= to the total number of linear equations) is given by

$$\tfrac{1}{2}(d-1)(d-2)+(d-3)+1=\tfrac{1}{2}(d-2)(d+1)$$

which is exactly the number of independent unknowns to determine C_{d-2} (see Table 1 of Section 2). Next, counting the number of points of intersection of C_d and C_{d-2}

$$(d-1)(d-2)+d-3+1=(d-2)d=C_{d-2}\cdot C_d$$

satisfying Bezout. For details of the applicability of Bezout's theorem with respect to *infinitely near* singularities, see [Abhyankar '73]. Then computing the $\mathrm{Res}_x(C_d, C_{d-2})$ which yields a polynomial of degree $d(d-2)$ in y and dividing by the common factors corresponding to the $(d-3)$ simple points (a polynomial of degree $(d-3)$ in y) and $\frac{1}{2}(d-2)(d-1)$ double points (a polynomial of degree $(d-2)(d-1)$ in y) yields a polynomial in y and t which is linear in y (for the single variable point) and thus gives a rational parameterization of y in terms of t. Similarly repeating with $\mathrm{Res}_y(C_d, C_{d-2})$ yields a rational parameterization of x in terms of t.

As a final example consider the *m-fold* cusp y^2-x^{2m+1} once again. We know from Section 2 that it is a rational curve with genus 0 and with a distinct double point and $m-1$ infinitely near double points at the origin (0, 0, 1) and a distinct $(2m-1)$-fold singularity and $m-1$ infinitely near double points at infinity (0, 1, 0). Now we pass a pencil of curve C_{2m-1} of degree $2m-1$ appropriately (as explained above) through these singularities and also through $2m+1-3=2m-2$ simple points of the *m*-fold cusp C_{2m+1}.

In the following let $F(X, Y, Z)=0$ be the equation of C_{2m+1} and $G(X, Y, Z)$ the equation of C_{2m-1}. Now the conditions available to specify a pencil of curves C_{2m-1} is given as follows. A total of $2m-2$ conditions are given by equating F and G at the $2m-2$ simple points of C_{2m+1}. Further by equating F and G and the corresponding transformed F_{T_i} and G_{T_i} (transformed by a sequence of quadratic transformations) at the *distinct* and *infinitely near* double points of the origin (0, 0, 1) and *infinitely near* double points of infinity (0, 1, 0). This totall accounts for $m+m-1=2m-1$ additional conditions. Finally through the $(2m-1)$-fold singularity at infinity of C_{2m+1} the pencil C_{2m-1} is passed with multiplicity $2m-2$ which is obtained by equating the equations and the partial derivatives $F_{X^jY^k}=G_{X^jY^k}$ for all $0\le j+k<2m-2$ which yields $\frac{1}{2}(2m-2)(2m-1)$ conditions. One final condition is achieved by equating one of the coefficients of C_{2m-1} to 't'. Hence totally the conditions available to specify the pencil of curves C_{2m-1} is given by

$$1+2m-2+2m-1+\tfrac{1}{2}(2m-2)(2m-1)=\tfrac{1}{2}(2m-1)(2m+2)$$

which is exactly the number of conditions required to specify a pencil of curve C_{2m-1} as given by Table 1 in Section 2. This then yields a linear system of $(2m-1)(m+1)$ equations in the same number of unknowns and can be easily solved.

Finally, note that the total number of intersections (counting multiplicities) between C_{2m-1} and C_{2m+1} are given by 1 {single variable point} + $(2m-2)$ {fixed simple points} + $2(2m-1)$ {double points} + $(2m-1)(2m-2)$ {$2m-2$ multiplicity of C_{2m-1} at the $(2m-1)$-fold singularity of C_{2m+1}} = $(2m-1)(2m+1)$ satisfying Bezout. Hence on computing the $\mathrm{Res}_x(C_{2m+1}, C_{2m-1})$ and dividing by the common factors corresponding to the $(2m-2)$ simple points, $(2m-1)$ double points and the $2m-2$ multiplicity of C_{2m-1} at the $(2m-1)$-fold singularity of C_{2m+1} yields a polynomial in y and t which is linear in y (for the single variable point) and thus gives a rational parameterization of y in terms of t. Similarly repeating with $\mathrm{Res}_y(C_{2m+1}, C_{2m-1})$ yields a rational parameterization of x in terms of t.

5. Conclusion

In this paper we presented algorithms to obtain rational parameterizations of irreducible algebraic curves of genus 0. These methods also apply to all irreducible *planar* algebraic curves, where planar curves are either specified by a single polynomial equation in the plane, $f(x, y)=0$ or may be specified by two polynomial equations in space, $f(x, y, z)=0$ and $g(x, y, z)=0$ (defining an irreducible space curve) where one of the two surface equations is rational. In the latter case the two equations specifying the space curve are easily mapped to a single polynomial equation $h(s, t)=0$ describing the curve in the parametric plane s–t of the rational surface. This mapping between the (x, y, z) points of the space curve and the (s, t) points of the plane curve is birational (almost one-to-one and onto) and hence a rational parameterization of this plane curve gives a rational parameterization of the space curve. Rational parameterization algorithms for surfaces thus provide this birational mapping between intersection curves in space and plane curves. For algorithms to parameterize low degree rational surfaces see [Abhyankar & Bajaj '87a, b], [Sederberg '87]. Rational parameterization techniques for irreducible algebraic space curves which are specified by two polynomial equations in space, without conditions on the rationality of the defining surfaces, are considered in [Abhyankar & Bajaj '87c].

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