Exercise 1: Algebraic Curve, Surface Splines

CS384R, CAM 395T: Fall 2008

September 9, 2008, Due: September 19, 2008

Question 1. The singularities of an algebraic plane curve $f(x, y) = 0$ are given by all solutions of $f(x, y) = f_x = f_y = 0$ where $f_x = \frac{\partial f(x,y)}{\partial x}$ and $f_y = \frac{\partial f(x,y)}{\partial y}$ are the partials of f with respect to x and y respectively.

a Compute the singularities of the following curves:

$$\begin{aligned} x^2 - y^7 &= 0 \\ x^2 - y^3 - y^7 &= 0 \\ x^3 + y^3 - 1 &= 0 \\ 2x^4 - 3x^2y + 2y^3 + y^4 &= 0 \end{aligned}$$

b Which of the curves in (a) are rational?

c Derive a parametric form

$$\begin{aligned} x(t) &= F(t) \\ y(t) &= G(t) \end{aligned}$$

for each of the curves, attempting to derive the simplest polynomial or rational form for F and G, whenever possible.

Question 2. An algebraic hyperelement is an $(n - 1)$ dimensional set of points defined in n-dimensional space by a single multivariate polynomial equation $f(x_0, x_1, \ldots x_{n-1}) = 0$ on n variables. For example a plane algebraic curve given by $f(x_0, y_0) = 0$ is also a 2-dimensional hyperelement and an algebraic surface $f(x_0, y_0, z_0)$ is a 3-dimensional hyperelement. Give a constructive proof that any quadratic hyperelement $f(x_0, x_1, \ldots x_n) = 0$ is rational, i.e. derive the rational parameterization.

{ Hint: Assume you have a point on the hyperelement and take a 1-dimensional family of lines through that point. }

Question 3. An algebraic space curve segment C is a connected piece of an algebraic space curve. C can be represented by a pair of algebraic surfaces, alongwith a pair of vertices (a pair of points on the curve), denoted as the starting vertex v_1 and an ending vertex v_2, of the curve segment. Additionally, a vector t_1 is provided, which is tangent to the curve segment C at v_1, and specifies the points of the curve segment from v_1 to v_2. Compute the intersection of a given surface $S : x^2 + y^2 + z^2 - 1 = 0$ with a space curve segment C given by the pair of surfaces $(x^2 + y^2 - z = 0, x = 0)$, and a starting vertex $v_1 = (0, 2, 4)$ and an ending vertex $v_2 = (0, -2, 4)$. Furthermore, the tangent vector $t_1 = (0, 1, -4)$ at v_1 is also given.

Question 4. A surface patch is a surface with boundary. It is also defined to be of finite area piece and therefore possesses boundary B_c which are cycles of affine curves segments lying on the surface. An algebraic surface patch P shall thus be represented by a single polynomial equation, and closed cycles of algebraic space curve segments lying on the surface. The algebraic surface patch points are defined to be to the left of the algebraic curve segment cycles, when viewed from the space which contains the normal of the surface and when traversing the boundary cycles in counter-clockwise order. Compute the intersection of a given spherical surface patch P with (a) the plane $y = z$ and (b) the surface $y^2 + z^2 - 1 = 0$. The patch P is given by $x^2 + y^2 + z^2 - 1 = 0$, and boundary space curve segment cycle specified by an ordered cycle of vertices $(1, 0, 0)$, $(0, -1, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$ lying on $x^2 + y^2 - 1 = 0$. Your answer should be represented as a collection of algebraic curve segments (as defined in Question 3).

Question 5. Consider an arrangement (collection) of spheres of varying radii in R^3 (atoms of a molecule). Each sphere in the arrangement is described by a 4-tuple (center-coordinates, radius). Given an arrangement of four spheres $\{ (0, 0, 0, 1), (0, 0, 1, 0.75), (0, 1, 0, 0.75), (1, 0, 0, 0.25)\}$ compute a boundary patch representation of the union of the arrangement (spatial description of the molecule), enumerating the various patch descriptions. Do this in the most efficient manner possible and describe your computation method. { Hint: Pairwise spheres and triple-wise sphere intersections need to be computed.}