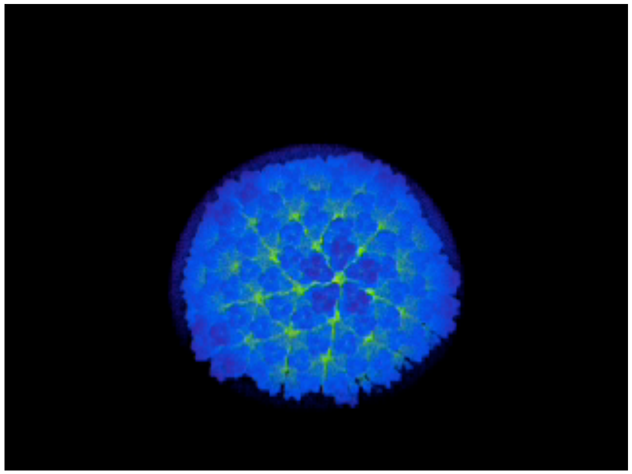
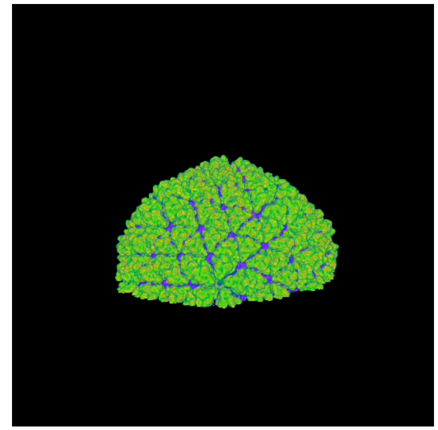
Geometric Modeling and Visualization

http://www.cs.utexas.edu/~bajaj/cs384R08/





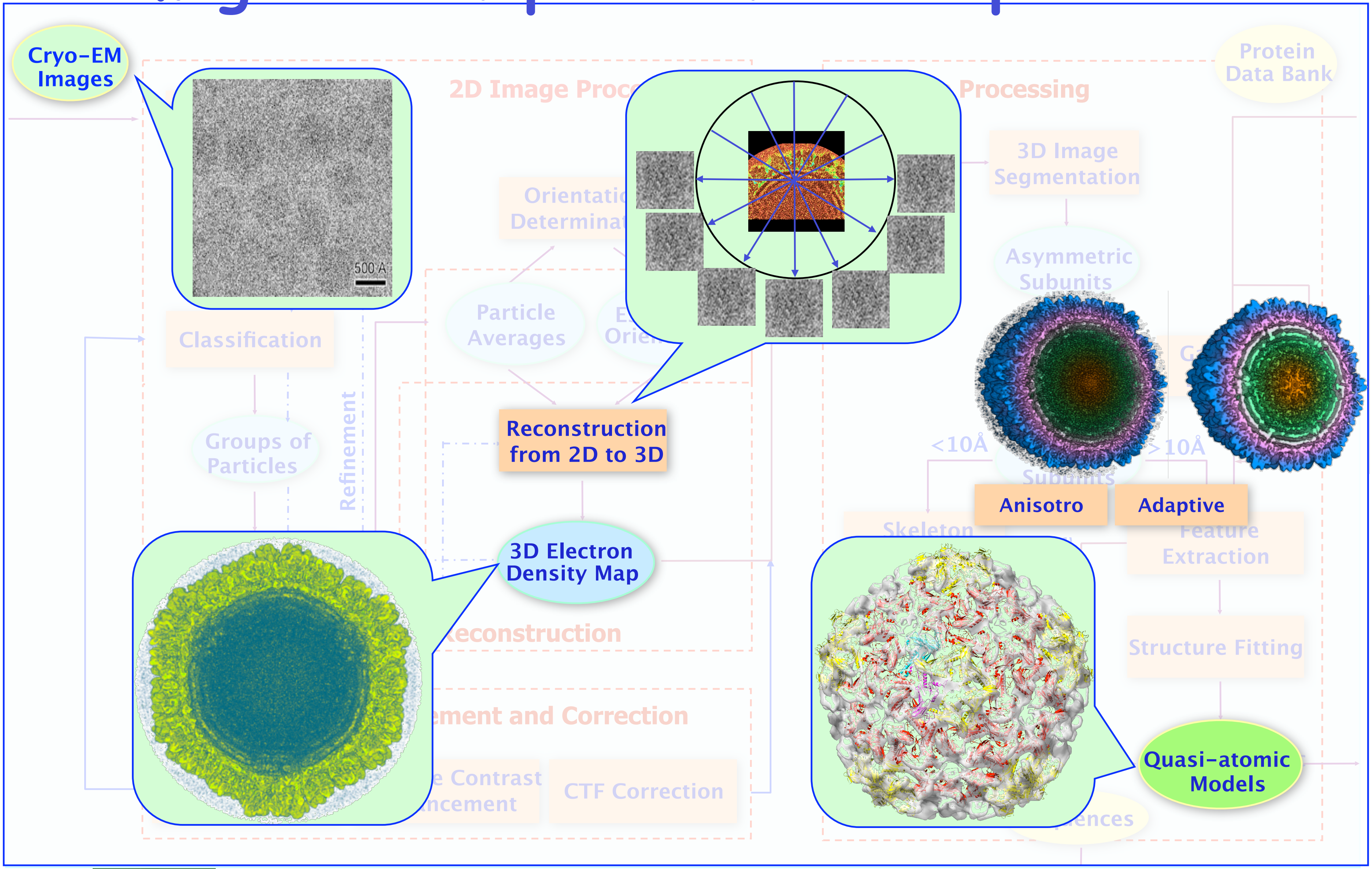
Lecture 2

Maps & Models: Discrete and Analytic Shape Representations

Computational Visualization Center
Institute of Computational and Engineering Sciences
Department of Computer Sciences
University of Texas at Austin

September 2008

Images to Maps to Models Pipeline - I



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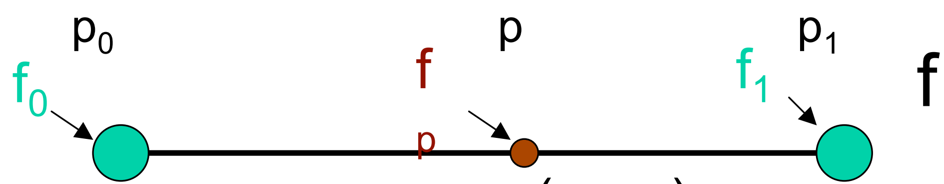
Linear Interpolation on a line segment



The Barycentric coordinates $\alpha = (\alpha_0\ \alpha_1)$ for any point p on line segment $<p_0\ p_1>$, are given by

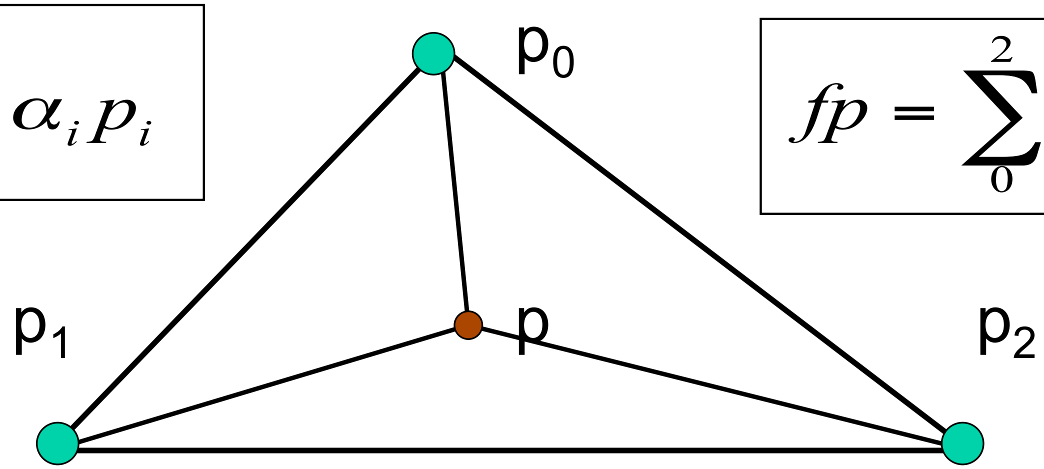
$$\alpha = \left(\frac{dist(p, p_1)}{dist(p_0, p_1)}, \frac{dist(p_0, p)}{dist(p_0, p_1)}\right)$$

which yields $\boxed{p = \alpha_0\ p_0 + \alpha_1\ p_1}$

and $\boxed{f_p = \alpha_0\ f_0 + \alpha_1\ f_1}$

Linear interpolation over a triangle

$$p = \sum_{0}^{2} \alpha_i p_i$$



$$fp = \sum_{0}^{2} \alpha_i fp_i$$

For a triangle p_0, p_1, p_2, the Barycentric coordinates $\alpha = (\alpha_0\ \alpha_1\ \alpha_2)$ for point p,

$$\alpha = \left(\frac{area(p, p_1, p_2)}{area(p_0, p_1, p_2)}, \frac{area(p_0, p, p_2)}{area(p_0, p_1, p_2)}, \frac{area(p_0, p_1, p)}{area(p_0, p_1, p_2)}\right)$$

Linear interpolant over a tetrahedron

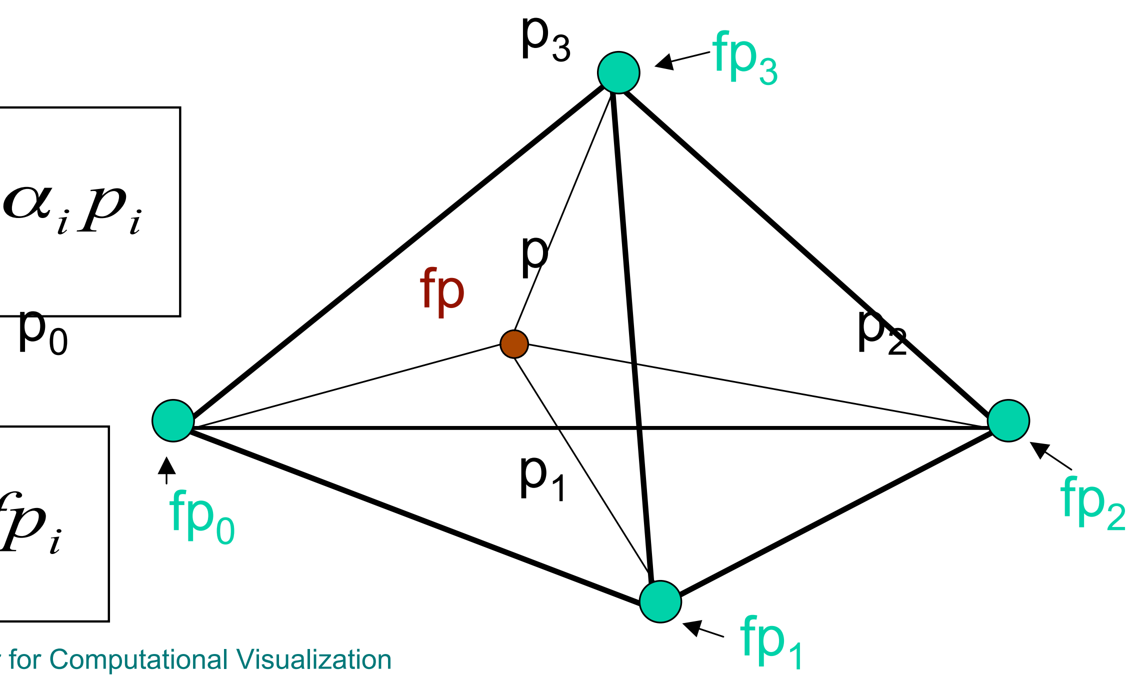
Linear Interpolation within a

- Tetrahedron (p_0,p_1,p_2,p_3)

  $\alpha = \alpha_i$ are the barycentric coordinates of p

$$p = \sum_{0}^{3} \alpha_i p_i$$

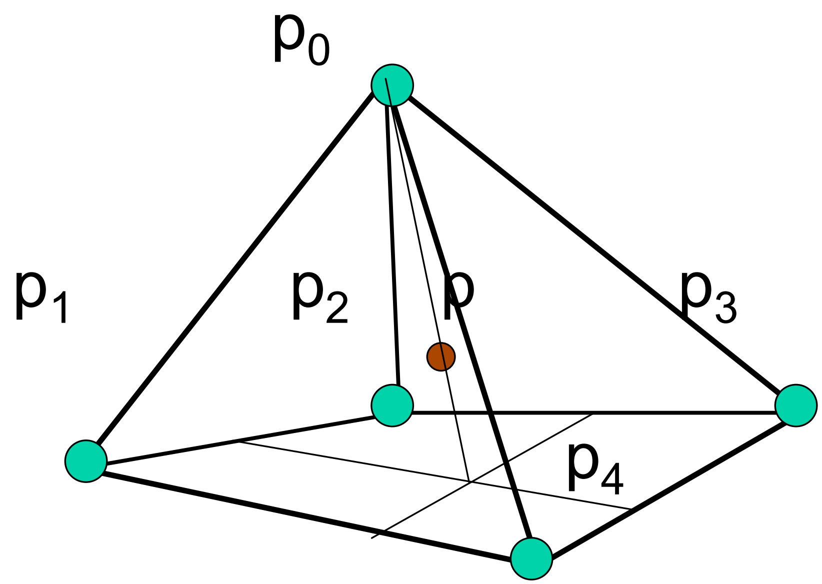
$$fp = \sum_{0}^{3} \alpha_i fp_i$$



Other 3D Finite elements

- Unit Pyramid (p_0, p_1, p_2, p_3, p_4)

$$p = up_0 + (1-u)(t(sp_1 + (1-s)p_2) + (1-t)(sp_3 + (1-s)p_4))$$



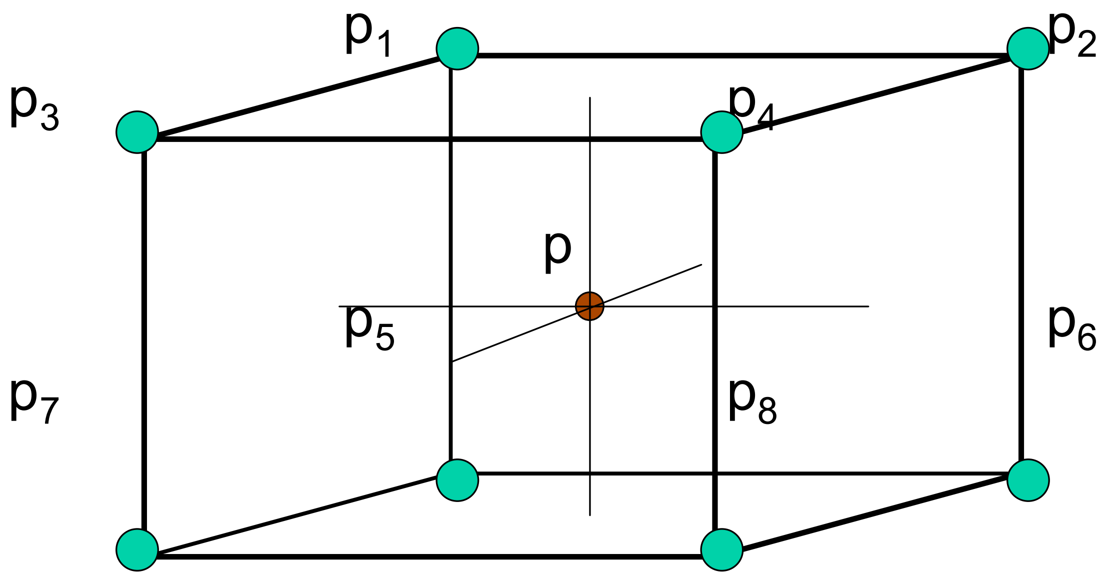
Note:
nonlinear

Other 3D Finite Elements

- Unit Cube ($p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8$)
  - Tensor in all 3 dimensions

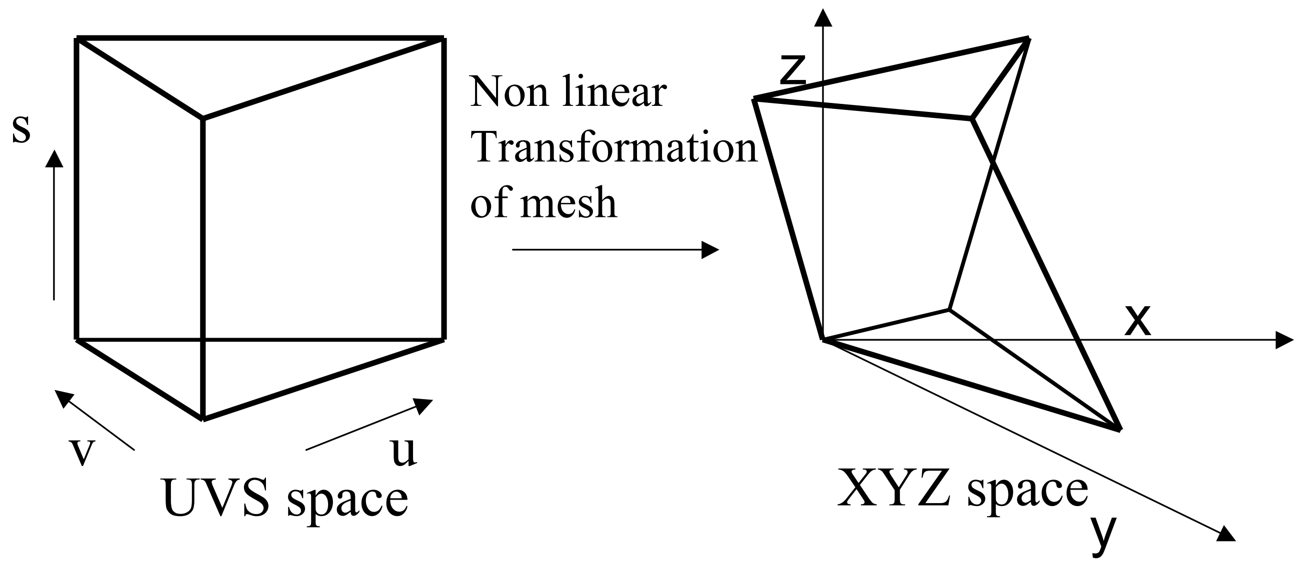
$$p = u(t(sp_1 + (1-s)p_2) + (1-t)(sp_3 + (1-s)p_4)) + (1-u)(t(sp_5 + (1-s)p_6) + (1-t)(sp_7 + (1-s)p_8))$$

Trilinear interpolant



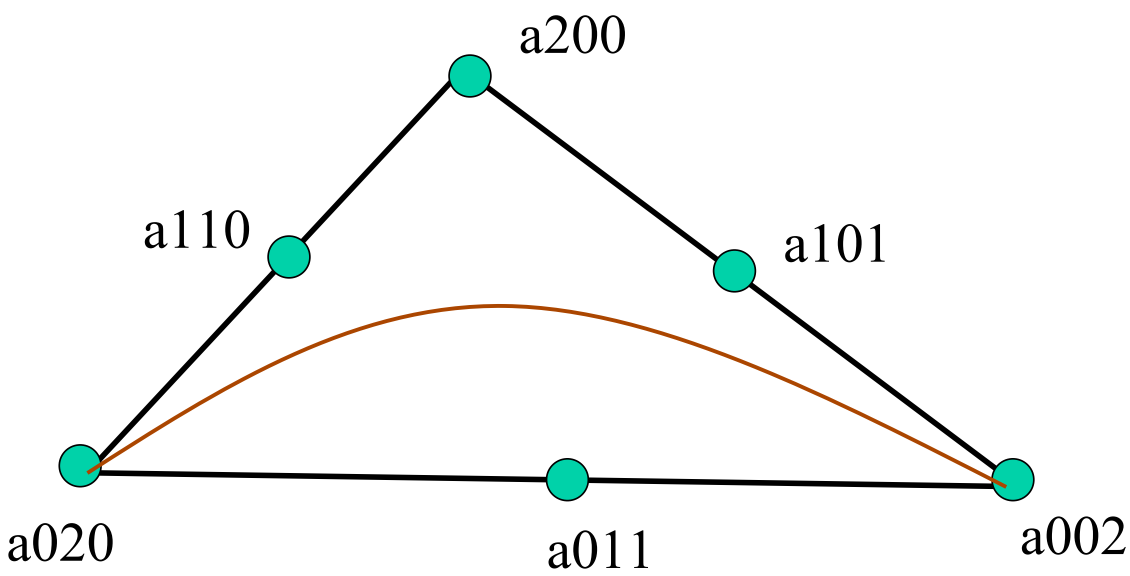
Non-linear finite elements-3d

- Irregular prism
  - Irregular prisms may be used to represent data.



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = ((1-u-v)\overrightarrow{\mathbf{p}_0} + u\overrightarrow{\mathbf{p}_1} + v\overrightarrow{\mathbf{p}_2})(1-s) + ((1-u-v)\overrightarrow{\mathbf{p}_3} + u\overrightarrow{\mathbf{p}_4} + v\overrightarrow{\mathbf{p}_5})(s)$$

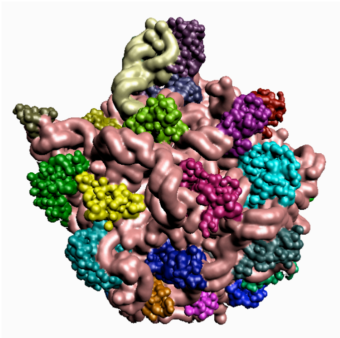
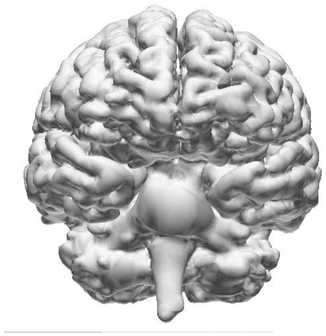
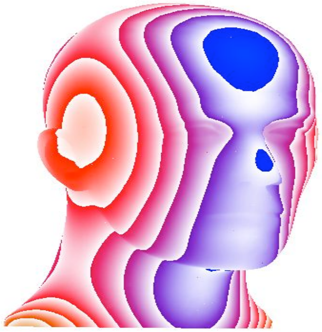
Non-Linear Algebraic Curve and Surface Finite Elements ?



The conic curve interpolant is the zero of the bivariate quadratic polynomial interpolant over the triangle

Algebraic Curve, Surface Splines

We shall consider the modeling of domains and function fields using algebraic splines

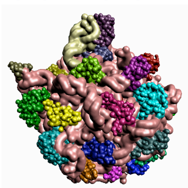
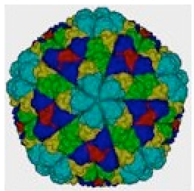
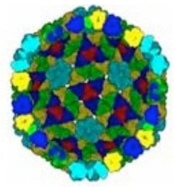
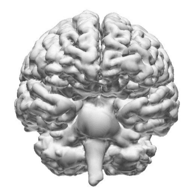
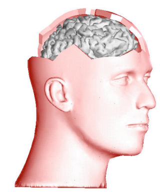
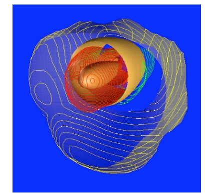
   

Algebraic Splines are a complex of piecewise :

- algebraic plane & space curves
- algebraic surfaces

So what are Algebraic Splines, again ?

Collection (Complex) of smooth finite elements of polynomial (algebraic) curves and surfaces with prescribed order of continuity between the finite elements.

1. The splines are variously called Simplex, Box, Polyhedral depending on the support of the polynomial pieces.
2. The splines also can variously use the B-basis (B stands for Basis) or the BB-basis (BB stands for Bernstein-Bezier), or the C-basis (C for Chebyshev), etc. depending on the choice of polynomial basis
3. B-Splines (E.g. UBs or NUBs) or B-patches or Rational B-splines (e.g. NURBs) or T-Splines or X-splines etc. are just several examples of polynomial splines which are rational.

Brief History of Algebraic Splines

1. A-Splines:
   - T-PACs, Cubics [Sederberg('98),Patterson-Paluzny('99)]
   - C^k A-splines within triangles [Bajaj,Xu('99)]
   - Regular A-splines over rectangular domains [Xu,Bajaj ('01)]
   - A-splines in Data Fitting [Bajaj,Xu('03)]
2. A-Patches:
   - C^1 piecewise quadric patches [Dahmen ('89)]
   - Clough-Tocher split for C^1 cubic patches [Guo ('91]
   - Single valued cubic C^1 A-patches [Bajaj, Chen, Xu ('95)]
   - Quintic C^2 A-patches [Bajaj, Xu ('97)]
   - Rational C^1 A-patches [Xu, Bajaj ('01)]
   - C^1 Prism A-patches and shell A-patches [Bajaj, Xu ('02,'03)]

C^k Triangular A-Splines

An A-spline element of degree d over the triangle $[p^1 p^2 p^3]$ is defined by

$$G_d(x, y) := F_d(\alpha) = F_d(\alpha_1, \alpha_2, \alpha_3) = 0$$

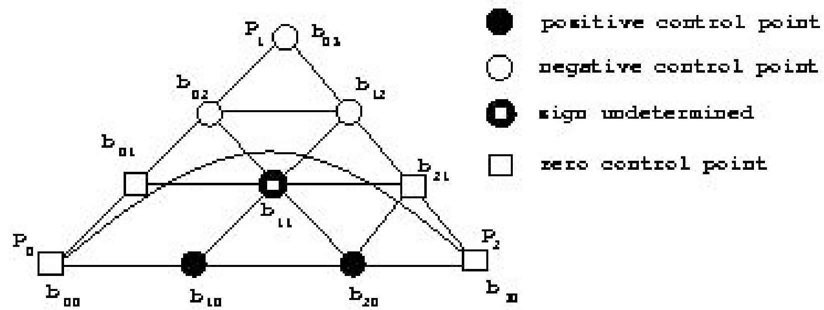
where

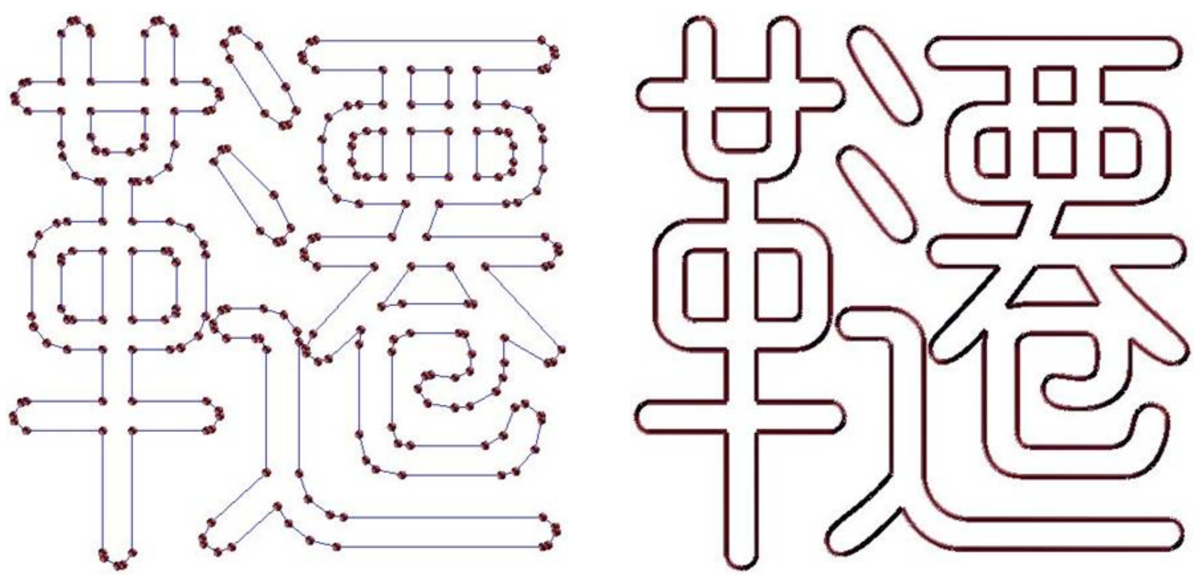
$$F_d(\alpha_1, \alpha_2, \alpha_3) = \sum_{i+j+k=d} b_{ijk} B^d_{ijk}(\alpha_1, \alpha_2, \alpha_3)), B^d_{ijk}(\alpha_1, \alpha_2, \alpha_3)) = \frac{d!}{i!j!k!} \alpha_1^i \alpha_2^j \alpha_3^k$$

and $(x, y)^T$ and $(\alpha_1, \alpha_2, \alpha_3)^T$ are related by

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

C^1 Cubic Triangular A-Spline





Rational Parametric Form of A-Splines

and in parametric spline form

$$X(t) = \sum_{i=0}^{d} w_i B_i^d(t) b_i / \sum_{i=0}^{d} w_i B_i^d(t), \;\; t \in [0, 1]$$

where $b_i \in R^3, w_i \in R$ and $B_i^d(t) = \{d!/[i!(d-i)!]\} t^i (1-t)^{d-i}$

- A-Splines: Local Interpolation and Approximation Using G^k-Continuous Piecewise Real Algebraic Curves *Computer Aided Geometric Design, (1999)*

C^k A-Patches

A-Patches are surface finite elements.
A-Patch element of degree d over the tetrahedron p_1, p_2, p_3, p_4 is defined by

$$G_d(x, y, z) := F_d(\alpha) = F_d(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0$$

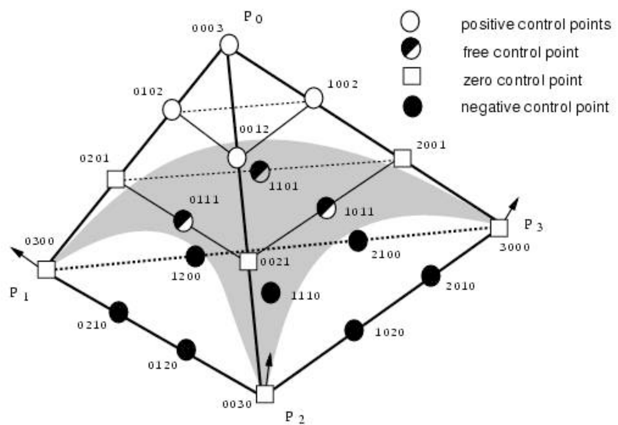
where

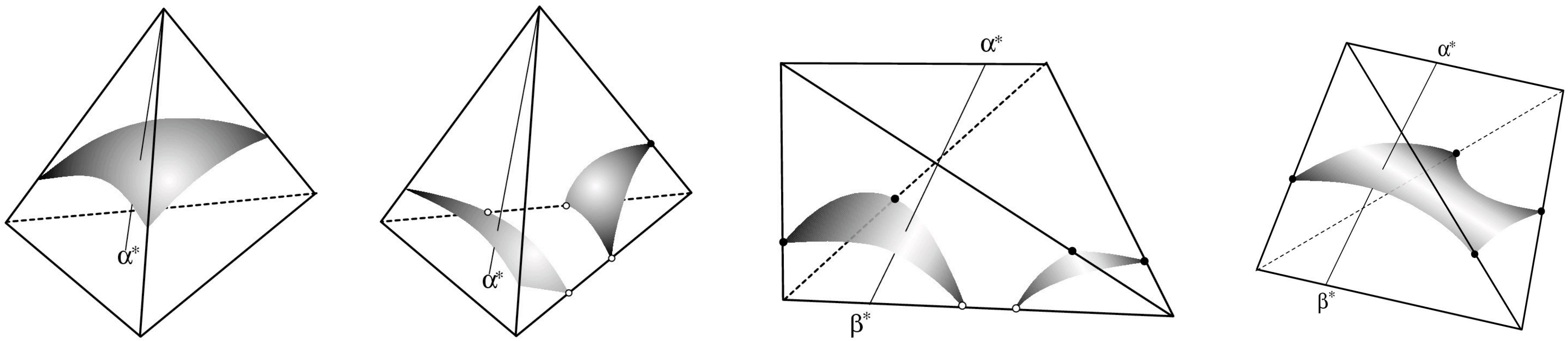
$$F_d(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \sum_{i+j+k+l=d} \alpha_{ijkl} B^d_{ijkl}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

and
$(x, y, z)^T$ and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$ are related by

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

Cubic A-patches on Tetrahedral Domains





- C^1 Modeling with Cubic A-patches *ACM Transactions on Graphics, 1995*
- C^1 Modeling with A-patches from Rational Trivariate Functions *Computer Aided Geometric Design, (2001)*

Prism C^1 A-patches

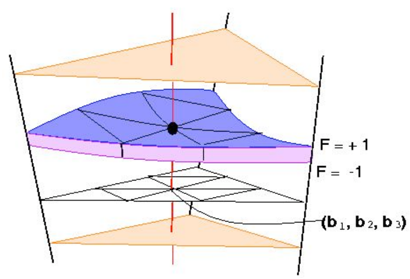
- Low degree algebraic surface finite element with dual implicit and rational parametric representations.
- The A-patch element is defined within a prism scaffold. For each triangle $v_i v_j v_k$ of a triangulation of the molecular surface, let

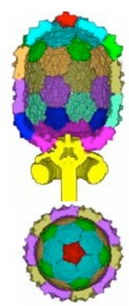
$$v_l(\lambda) = v_l + \lambda n_l, \qquad l = i, j, k$$

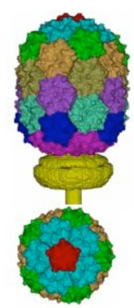
Define the prism

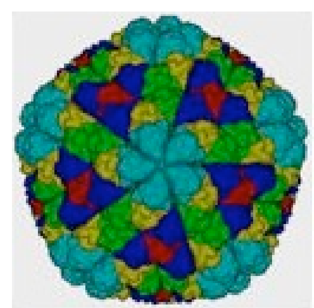
$$D_{ijk} := \{p : \ p = b_1 v_i(\lambda) + b_2 v_j(\lambda) + b_3 v_k(\lambda), \ \lambda \in I_{ijk}\}$$

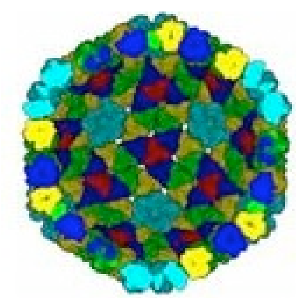
where (b_1, b_2, b_3) are the barycentric coordinates of points in $v_i v_j v_k$.











Hierarchical Multiresolution Reconstruction of Shell Surfaces
Computer Aided Geometric Design, (2002)

Can we convert between Algebraic Splines and Parametric Splines ?

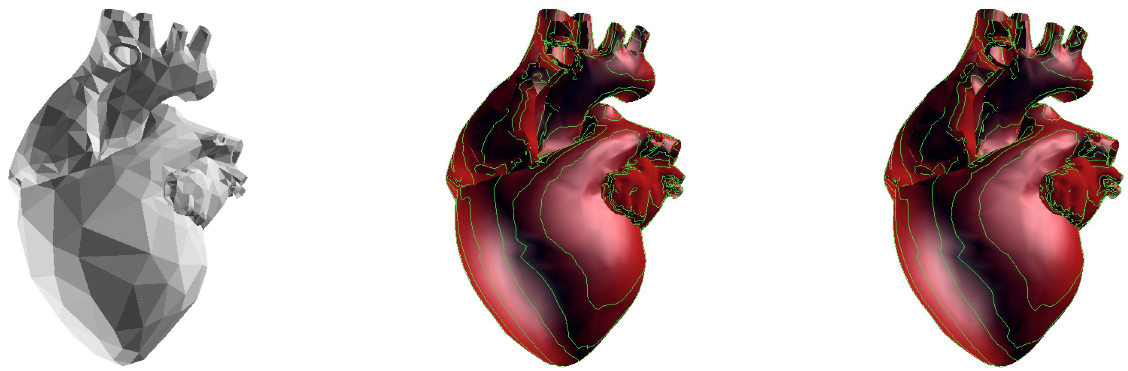


Figure: C^1 Rational Algebraic Splines

Answer: Since the algebraic plane/space curve and/or algebraic surface in general are not rational we need to construct rational parametric spline *approximations. !*

NURBs Approximation of A-splines and A-patches International Journal of Computational Geometry and Applications, (2003)

Additional Reading

- C. Bajaj, G. Xu
  A-Splines: Local Interpolation and Approximation Using Gk- Continuous Piecewise Real Algebraic Curves
  Computer Aided Geometric Design, 16:6(1999), 557-578. http://www.cs.utexas.edu/~bajaj/cs384R08/reading/ck-aspline.pdf
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- C.Bajaj "Free Form Modeling with Implicit Surface Patches (Chap 4 of Bloomenthal et al book)" http://www.cs.utexas.edu/~bajaj/cs384R08/reading/chap4-implicit.pdf
- C.Bajaj, G.Xu "Smooth Shell Construction with Mixed Prism Fat Surfaces" http://www.cs.utexas.edu/~bajaj/cs384R08/reading/smooth-shell.pdf