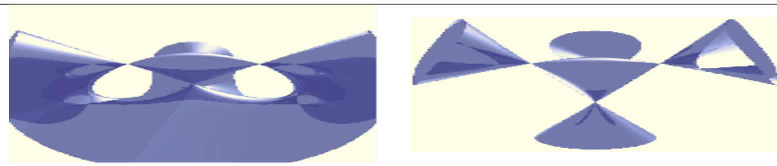
Geometric Modeling and Visualization

http://www.cs.utexas.edu/~bajaj/cs384R08/

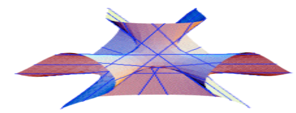
Example Algebraic Surfaces



The Clebsch Diagonal Cubic

$81*x^3+81*y^3+81*z^3-189*x^2*y-189*x^2*z-189*y^2*x-189*y^2*z-189*z^2*x-189*z^2*y+54*x*y*z+126*x*y+126*x*z+126*y*z-9*x^2,-9*y^2-9*z^2-9*x-9*y-9*z+1$

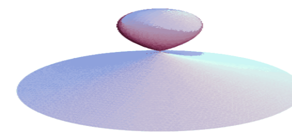
(27 real lines with 10 triple points)



The Cayley Cubic

$-5*x^2*y-5*x^2*z-5*y^2*x-5*y^2*z-5*z^2*y-5*z^2*x+2*x*y+2*x*z+2*y*z$

(9 real lines = 6 connecting 4 double points, and 3 in a coplanar config)



The Ding-Dong Surface

$x^2+y^2-(1-z)*z^2$

Lecture 3

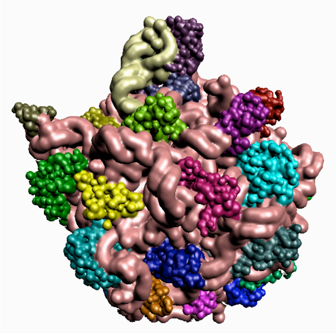
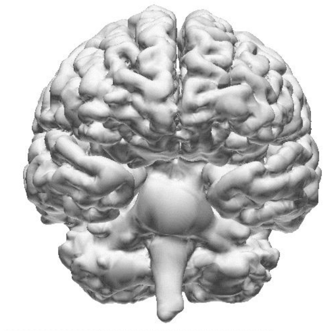
Algebraic Curves and Surfaces: Implicit and Parametric forms

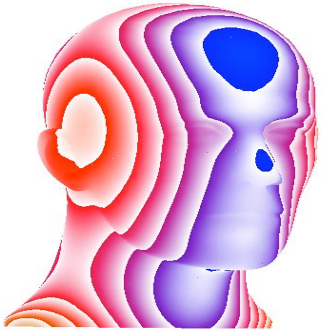
Computational Visualization Center
Institute of Computational and Engineering Sciences
Department of Computer Sciences
University of Texas at Austin

September 2008

Algebraic Curve, Surface Splines

We shall consider the modeling of domains and function fields using algebraic splines



Algebraic Splines are a complex of piecewise :

- algebraic plane & space curves
- algebraic surfaces

Algebraic Plane curves

- An algebraic plane curve in implicit form is a hyperelement of dimension 1 in R^2:

$$f(x, y) = 0 \tag{1}$$

- An algebraic plane curve in parametric form is an algebraic variety of dimension 1 in R^3. It is also a rational mapping from R^1 into R^2.

$$x = f_1(s)/f_3(s) \tag{2}$$
$$y = f_2(s)/f_3(s) \tag{3}$$

Algebraic Space curves

- An algebraic space curve can be implicitly defined as the intersection of two surfaces given in implicit form:

$$f_1(x, y, z) = 0 \; f_2(x, y, z) = 0 \tag{4}$$

- or alternatively as the intersection of two surfaces given in parameteric form:

$$(x = f_{1,1}(s_1, t_1), y = f_{2,1}(s_1, t_1), z = f_{3,1}(s_1, t_1)) \tag{5}$$
$$(x = f_{1,2}(s_2, t_2), y = f_{2,2}(s_2, t_2), z = f_{3,2}(s_2, t_2)) \tag{6}$$

  where all the $f_{i,j}$ are rational functions in s_i, t_j
- Rational algebraic space curves can also be represented as:

$$x = f_1(s), y = f_2(s), z = f_3(s) \tag{7}$$

  where the f_i are rational functions in s.

Parameterization of algebraic curves

Theorem An algebraic curve P is rational iff the Genus(P)= 0.

The proof is classical, though non-trivial. See also, Abhyankar's Algebraic Geometry for Scientists & Engineers *AMS Publications, (1990)*

Constructive proof, genus computation, and parameterization algorithm is available from:

Automatic Parameterization of Rational Curves and Surfaces III : Algebraic Plane Curves *Computer Aided Geometric Design, (1988)*

Parameterization of algebraic space curves ..1

For algebraic space curves C given as intersection of two algebraic surfaces there exists a birational correspondence between points of C and points of a plane curve P.

The genus of C is same as the genus of P.

Hence C is rational iff $Genus(P) = 0$.

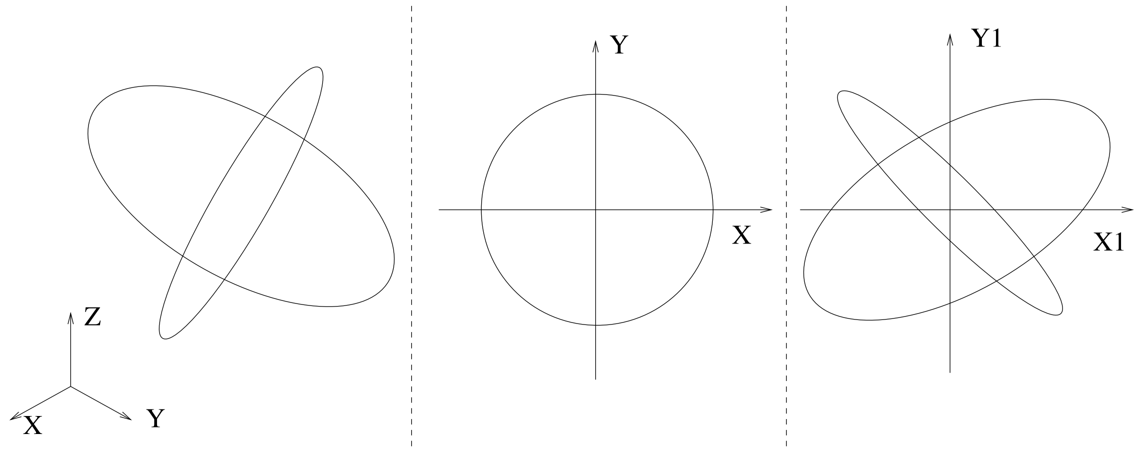
Algorithm :

- Construct a birationally equivalent plane curve P from C
- Generate a rational parametrization for P
- Construct a rational surface S containing C.

Automatic Parameterization of Rational Curves and Surfaces IV : Algebraic Space Curves *ACM Transactions on Graphics, (1989)*

Parameterization of algebraic space curves ..2

Given: Irreducible space curve $C = (f = 0 \cap g = 0)$, and f, g not tangent along C.
Compute: Project C to an irreducible plane curve P, properly, to yield a birational map from P to C.



1. Space curve C as intersection of two axis aligned cylinders

$$C : (f = z^2 + x^2 - 1 \cap g = z^2 + y^2 - 1) \tag{8}$$

2. Badly chosen projection direction results in P not birationally related to C

$$P : (x^2 + z^2 - 1)^2 = 0 \tag{9}$$

3. Birationally equivalent plane curve P with properly chosen projection direction

$$P : (8y_1^2 - 4x_1y_1 + 5x_1^2 - 9)(8y_1^2 + 12x_1y_1 + 5x_1^2 - 1) = 0 \tag{10}$$

Parameterization of algebraic space curves ..3

Projection can be computed using Elimination Theory. One way to eliminate a variable from two polynomials, is via Sylvester's polynomial resultant:

Given two polynomials

$$f(x) = a_m x^m + a_{m-1} x^{m-1} ... a_0 \tag{11}$$

$$g(x) = b_n x^n + b_{n-1} x^{n-1} ... b_0 \tag{12}$$

The Sylvester resultant matrix is constructed by rows of coefficients of f, shifted, followed by rows of coefficients of g, shifted.

To project along the z axis, write both equation as just polynomials in z, construct the matrix of coefficients in x, y, and the Sylvester resultant (projection) is the determinant.

Of course, the z axis may not be a proper projection direction. Hence first choose a valid transformation, to enable the projection to yield a rational (inverse) map.

Parameterization of algebraic space curves ..4

Choosing a valid projection direction:

Consider a general linear transformation to apply to f, g:

$$x = a_1x_1 + b_1y_1 + c_1z_1,\ y = a_2x_1 + b_2y_1 + c_2z_1,\ z = a_3x_1 + b_3y_1 + c_3z_1 \quad (13)$$

On substituting, we obtain the transformed equations

$$f_1(x_1, y_1, z_1) = 0,\ g_1(x_1, y_1, z_1) = 0$$

Compute Resultant $h(x_1, y_1)$ eliminating z_1 to yield the projected plane curve P: $h = 0$.

To obtain a birational inverse map $z_1 = H(x_1, y_1)$, which exists when the projection degree is 1, we need to satisfy:

- Determinant of linear transformation to be nonzero
- Equation h of projected plane curve P is not a power of an irreducible polynomial.

A random choice of coefficients for the linear transformation, works with high probability.

Algebraic surfaces

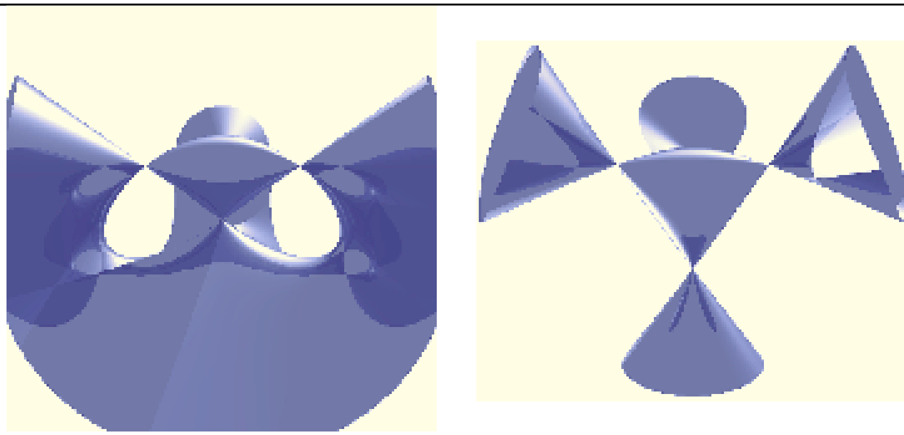
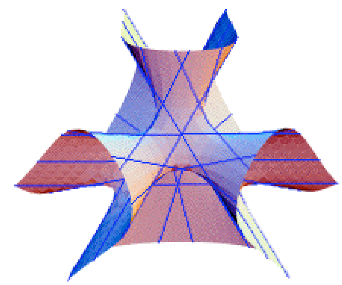
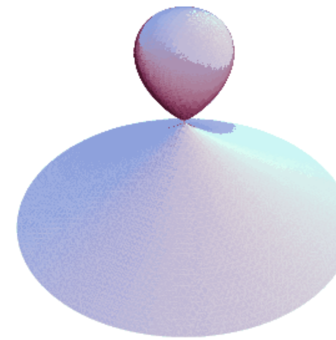
- An algebraic surface in implicit form is a hyperelement of dimension 2 in R^3:

$$f(x, y, z) = 0 \tag{14}$$

- An algebraic surface in parametric form is an algebraic variety of dimension 2 in R^5. It is also a rational mapping from R^2 into R^3.

$$x = f_1(s, t)/f_4(s, t) \tag{15}$$
$$y = f_2(s, t)/f_4(s, t) \tag{16}$$
$$z = f_3(s, t)/f_4(s, t) \tag{17}$$

Example Algebraic Surfaces

		
The Clebsch Diagonal Cubic	The Cayley Cubic	The Ding-Dong Surface
$81*x^3+81*y^3+81*z^3-189*x^2*y-189*x^2*z-189*y^2*x-189*y^2*z-189*z^2*x-189*z^2*y+54*x*y*z+126*x*y+126*x*z+126*y*z-9*x^2, -9*y^2-9*z^2-9*x-9*y-9*z+1$	$-5*x^2*y-5*x^2*z-5*y^2*x-5*y^2*z-5*z^2*y-5*z^2*x+2*x*y+2*x*z+2*y*z$	$x^2+y^2-(1-z)*z^2$
(27 real lines with 10 triple points)	(9 real lines = 6 connecting 4 double points, and 3 in a coplanar config)	

Cubic Algebraic Surfaces: Historical Gossip Column!

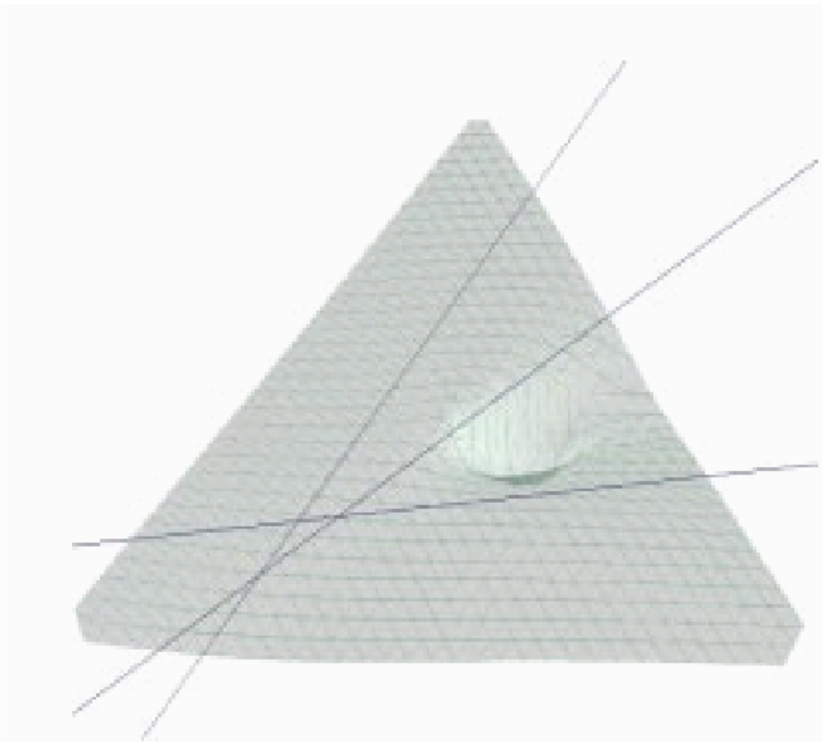
[1849 Cayley, Salmon] Exactly 27 straight lines on a general cubic surface

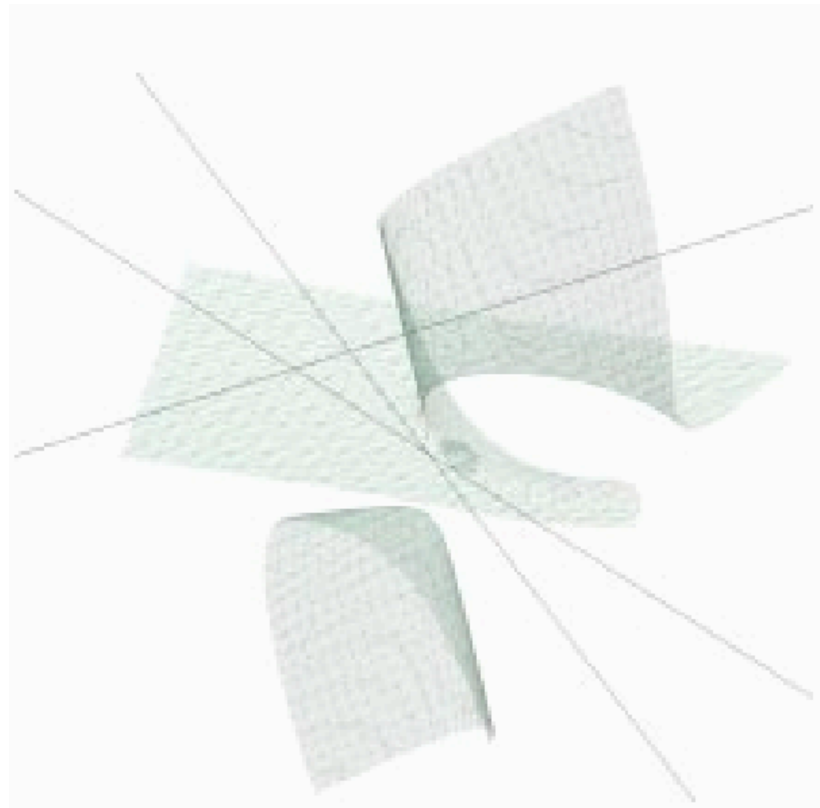
[1856 Steiner] The nine straight lines in which the surfaces of two arbitrarily given trihedra intersect each other determine together with one given point, a cubic surface.

[1858,1863 Schlafli] classifies cubic surfaces into 23 species with respect to the number of real straight lines and tri-tangent planes on them

[1866 Cremona] establishes connections between the 27 lines on a cubic surface and Pascals Mystic hexagram:- If a hexagon is inscribed in any conic section, then the points where opposite sides meet are collinear.

45 Tri-Tangents on Smooth Cubic Surfaces





Why are the 27 lines useful to geometric modeling ?

Given two skew lines on the cubic surface $f(x, y, z) = 0$

$$l_1(u) = \begin{bmatrix} x_1(u) \\ y_1(u) \\ z_1(u) \end{bmatrix} \text{ and } l_2(u) = \begin{bmatrix} x_2(u) \\ y_2(u) \\ z_2(u) \end{bmatrix}$$

One can derive the following surface parameterization :

$$P(u, v) = \begin{bmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{bmatrix} = \frac{al_1 + bl_2}{a + b} = \frac{a(u, v)l_1(u) + b(u, v)l_2(v)}{a(u, v) + b(u, v)}$$

where

$$a = a(u, v) = \nabla f(l_2(v)).[l_1(u) - l_2(v)]$$

$$b = b(u, v) = \nabla f(l_1(v)).[l_1(u) - l_2(v)]$$

Algorithm for Computing the 27 Lines

$$f(x,y,z) = \begin{array}{c} Ax^3 + By^3 + Cz63 + Dx62y + Ex^2z + \\ Fxy^2 + Gy^2z + Hxz^2 + Iyz^2 + Jxyz + kx^2 + \\ Ly^2 + Mz^2 + Nxy + Oxz + Pyz + Qx + Ry + Sz + T = 0 \end{array}$$

Through intersection with tangent planes, one can reduce this to

$$\hat{f}_2(\hat{x},\hat{y}) + \hat{g}_3(\hat{x},\hat{y}) = 0$$

With a generic parameterization of the singular tangent cubics, one derives a polynomial $P_{81}(t)$ of degree 81.

Properties of the polynomial $P_{81}(t)$

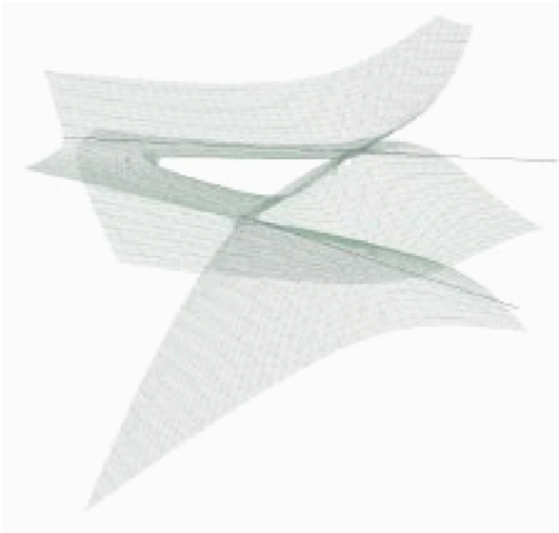
Theorem The polynomial $P_{81}(t)$ obtained by taking the resultant of $\hat{f}_2$ and $\hat{g}_3$ factors as $P_{81}(t) = P_{27}(t)[P_3(t)]^6[P_6(t)]^6$, where $P_3(t)$, and $P_6(t)$ are degree 3 and 6 respectively.

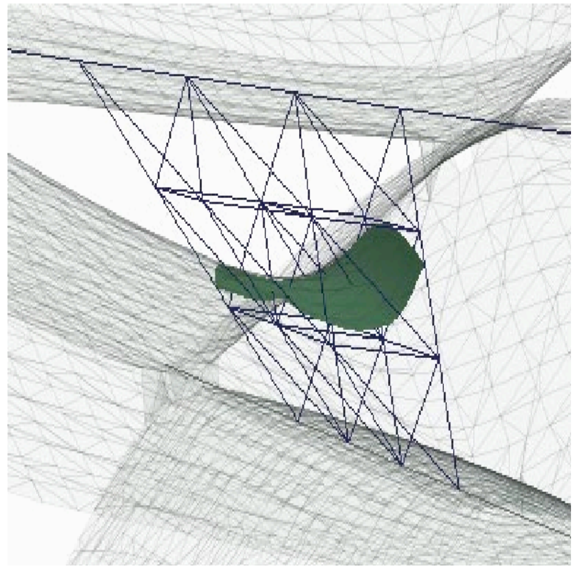
Theorem Simple real roots of $P_{27}(t) = 0$ correspond to real lines on the surface.

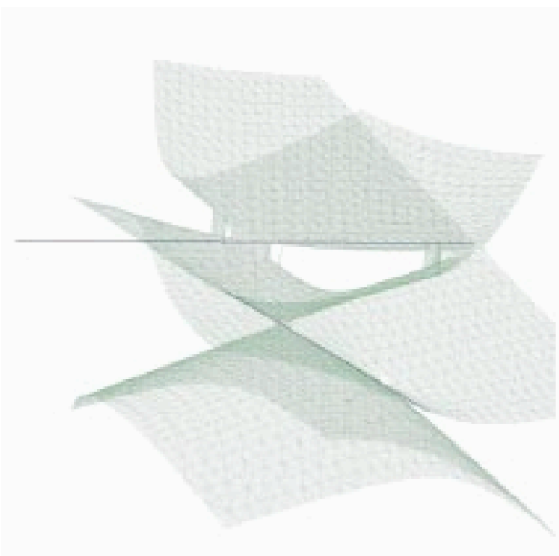
Proof and algorithm details available from

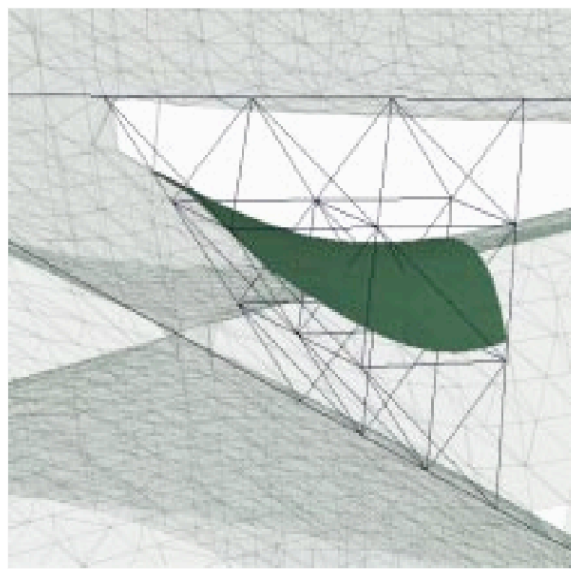
Rational parameterizations of non singular cubic surfaces *ACM Transactions on Graphics, (1998)*

Some Examples









Parameterization of algebraic surfaces

Theorem An algebraic surface S is rational iff the Arithmetic Genus(S)= Second Pluri-Genus (S) = 0.

The proof is attributed to Castelnuovo. See, Zariski's Algebraic Surfaces *Ergeb. Math. , Springer, (1935)*

Several examples of well known rational algebraic surfaces include: Cubic, Del Pezzo, Hirzebruch, Veronese, Steiner, etc.

What if the Algebraic Curve and/or Surface is Not Rational ?

Answer: Construct Rational Spline Approximations for a piecewise parameterization!

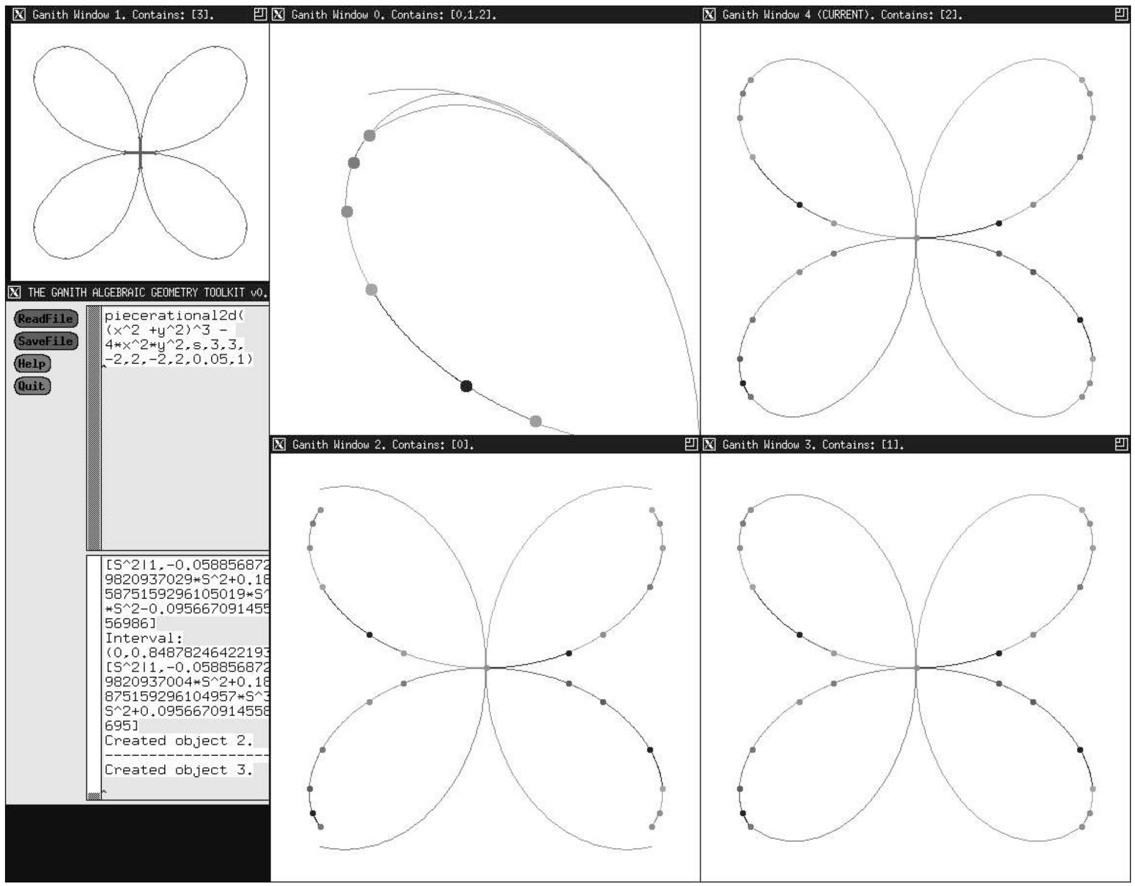
Rational Spline Approximation of Algebraic Plane Curves

Input: Given a real algebraic curve **C** of degree d, a bounding box B, a finite precision real number ϵ and integers m, n with $m + n \leq d$. The curve **C** within the bounding box B is denoted as $\mathbf{C}_B$.

Output: A C^{-1}, C^0 or C^1 continuous piecewise rational ϵ-approximation of all portions of **C** within the given bounding box B, with each rational function $\frac{P_i}{Q_i}$ of degree $P_i \leq m$ and degree $Q_i \leq n$ and $m + n \leq d$.

Piecewise Rational Approximation of Real Algebraic Curves *Journal of Computational Mathematics, (1997)*

Rational Spline Approximation of $(x^2 + y^2)^3 - 4x^2y^2 = 0$ in Ganith



2. Algorithm

- Compute the intersections, the singular points S and the x-extreme points T of $\mathbf{C}_B$.
- Compute Newton factorization (via Hensel lifting) for each (x_i, y_i) in S and obtain a power series representation for each analytic branch of $\mathbf{C}$ at (x_i, y_i) given by

$$\begin{cases} X(s) = x_i + s^{k_i} \\ Y(s) = \sum_{j=0}^{\infty} c_j^{(i)} s^j, \quad c_0^{(i)} = y_i \end{cases} \tag{18}$$

or

$$\begin{cases} Y(s) = y_i + s^{k_i} \\ X(s) = \sum_{j=0}^{\infty} \tilde{c}_j^{(i)} s^j, \quad \tilde{c}_0^{(i)} = x_i \end{cases} \tag{19}$$

3. Algorithm Contd.

- Compute $\frac{P_{mn}(s)}{Q_{mn}(s)}$ the (m, n) rational Padé approximation of $Y(s)$.
- Compute $\beta > 0$ a real number, corresponding to points $(\tilde{x}_i = X(\beta), \tilde{y}_i = Y(\beta))$ and $(\hat{x}_i = X(-\beta), \hat{y}_i = Y(-\beta))$ on the analytic branch of the original curve **C**, such that $\frac{P_{mn}(s)}{Q_{mn}(s)}$ is convergent for $s \in [-\beta, \beta]$.

4. Algorithm Contd.

- Modify $P_{mn}(s)/Q_{mn}(s)$ to $\tilde{P}_{mn}(s)/\tilde{Q}_{mn}(s)$ such that $\tilde{P}_{mn}(s)/\tilde{Q}_{mn}(s)$ is C^1 continuous approximation of $Y(s)$ on $[0, \beta]$,
- Denote the set of all the points $(\tilde{x}_i, \tilde{y}_i)$, $(\hat{x}_i, \hat{y}_i)$, the set T and the boundary points of $\mathbf{C}_B$ by V. Starting from each (simple) point (x_i, y_i) in V, $\mathbf{C}_B$ is traced out by the Taylor approximation

$$\begin{aligned} X(s) &= x_i + s \\ Y(s) &= \sum_{j=0}^{\infty} c_j^{(i)} s^j, \quad c_0^{(i)} = y_i \end{aligned}$$

5. Results

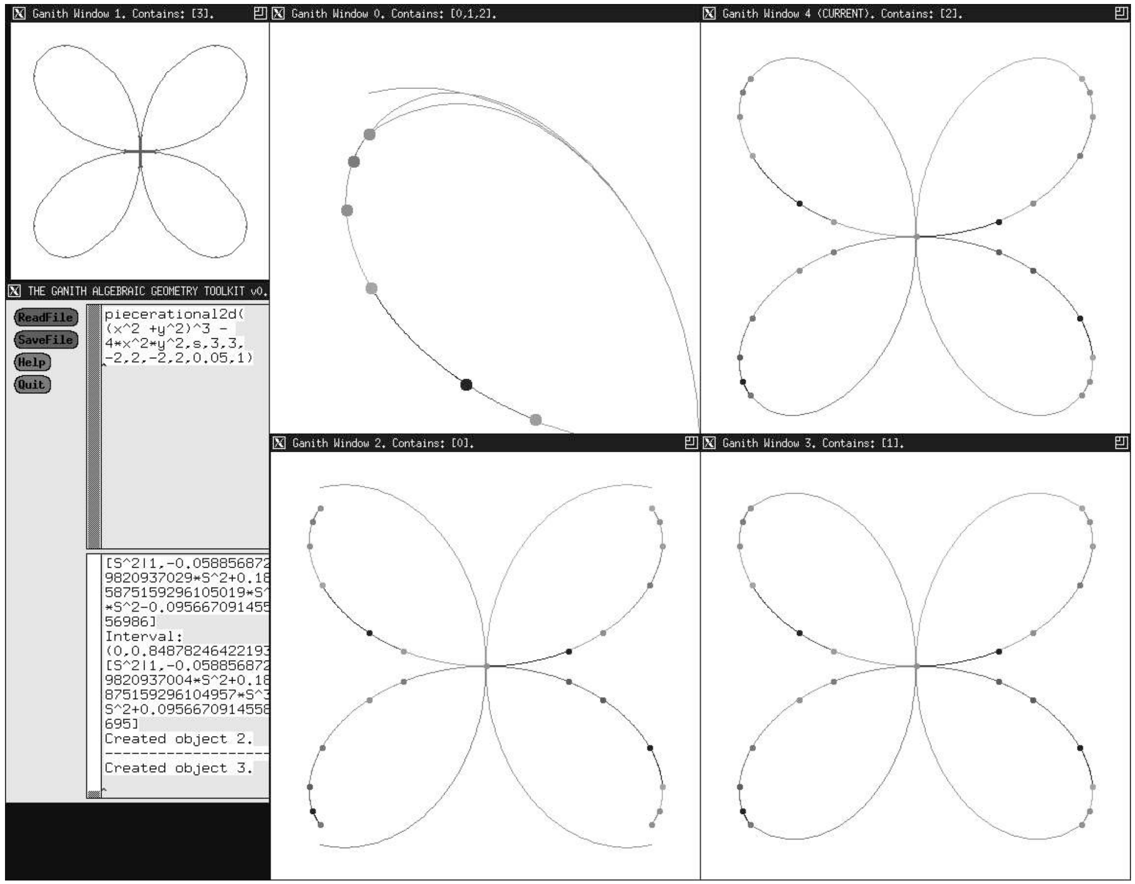


Figure: $(x^2 + y^2)^3 - 4x^2y^2 = 0$

Rational Spline Approximation of Space Curves

Given a real intersection space curve *SC* which is either the intersection of two implicitly defined surfaces $f_1(x, y, z) = 0,\ f_2(x, y, z) = 0$, or, the intersection of two parametric surfaces defined by

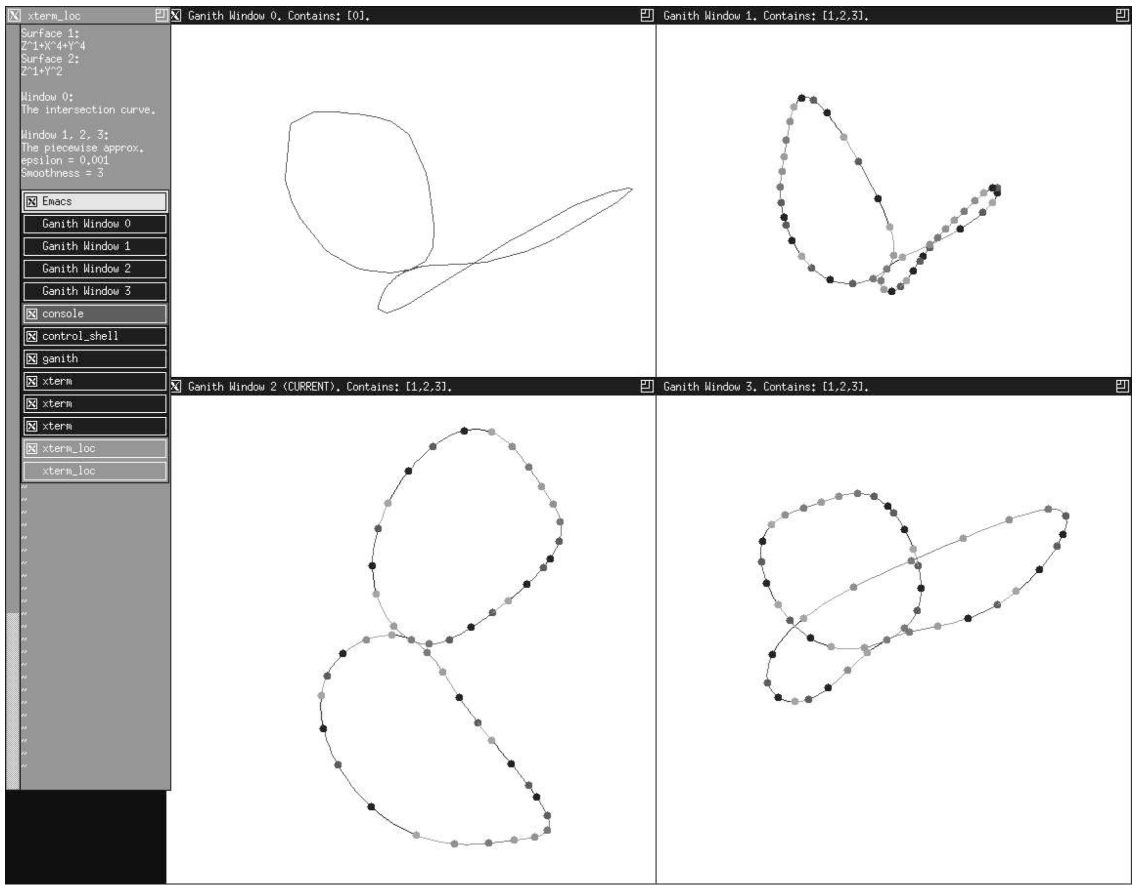
$$
\begin{aligned}
X_1(u_1, v_1) &= [G_{11}(u_1, v_1)\ G_{21}(u_1, v_1),\ G_{31}(u, v_1)]^T \\
X_2(u_2, v_2) &= [G_{12}(u_2, v_2)\ G_{22}(u_2, v_2),\ G_{32}(u_2, v_2)]^T
\end{aligned}
$$

within a bounding box B and an error bound $\epsilon > 0$, a continuity index k, construct a C^k (or G^k) continuous piecewise parametric rational ϵ-approximation of all portions of *SC* within B.

NURBS Approximation of Surface/Surface Intersection Curves
Advances in Computational Mathematics, (1994)

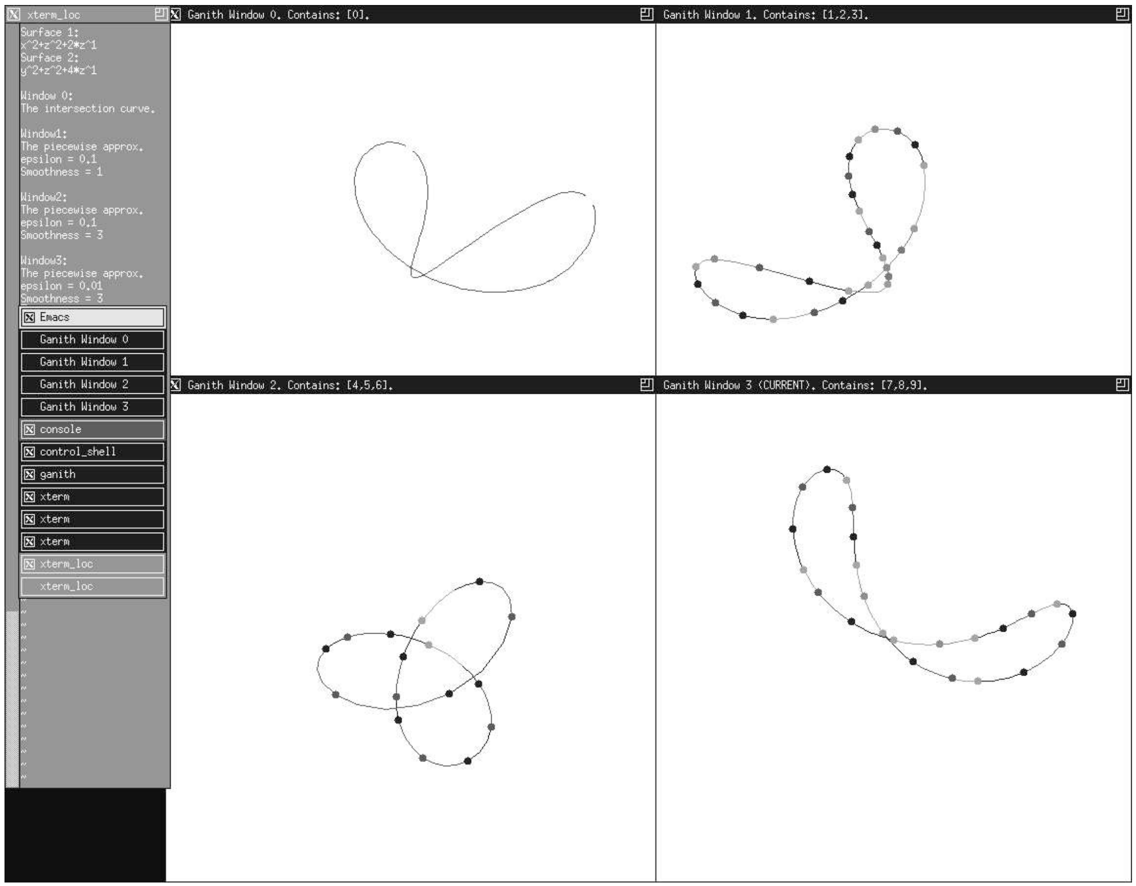
Results from Ganith - Intersection of Two implicit surfaces

Surfaces: $x^4 + y^4 + z = 0$ and $y^2 + z = 0$



Results from Ganith - Intersection of Implicit and Parametric Surfaces

Surfaces: $x^2 + z^2 + 2z = 0$ and $x = \frac{s+st^2}{1+t^2}, y = \frac{2-2t^2}{1+t^2}, z = \frac{4t-2-2t^2}{1+t^2}$



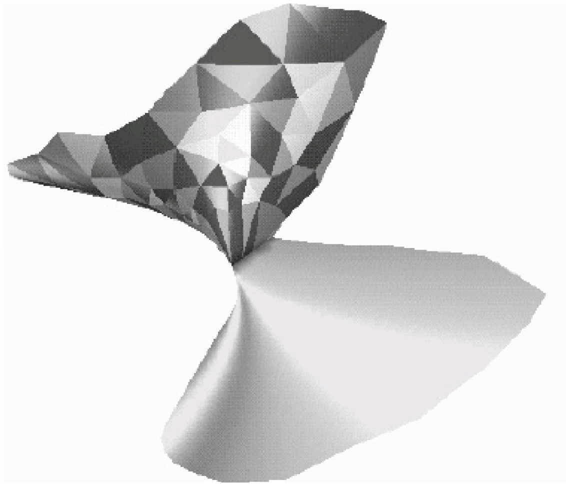
Rational Spline Approximation of Algebraic Surfaces

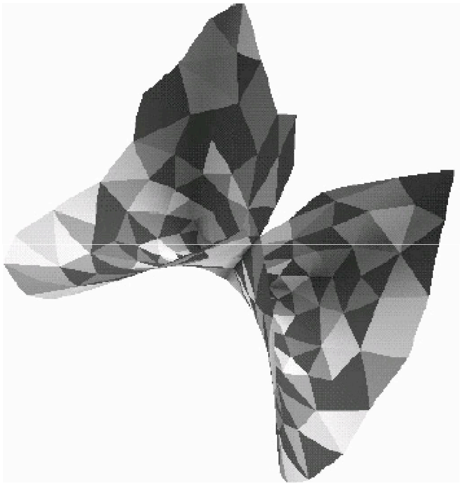
Given an implicit surface defined by a function $f(x, y, z) = 0$ and bounding box, create a piecewise rational spline approximation of the surface within the bounding box.

Spline Approximations of Real Algebraic Surfaces *Journal of Symbolic Computation, Special Isssue on Parametric Algebraic Curves and Applications, (1997)*

Results from Ganith

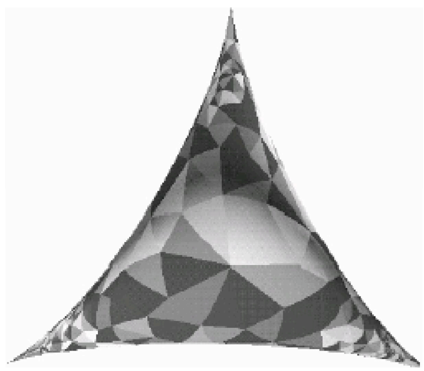
Cartan Surface: $f = x^2 - y * z^2 = 0$ has a singular point at $(0, 0, 0)$ and a singular line $(x = 0, z = 0)$.

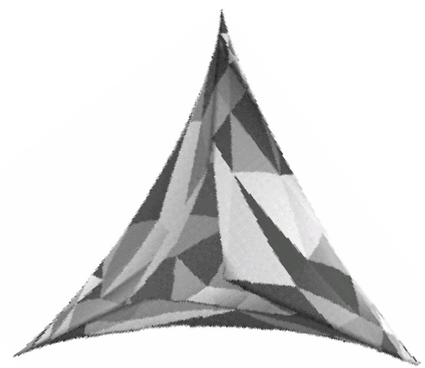




Results from Ganith

Patch of a **Steiner Surface:**
$f = x^2 * y^2 + y^2 * z^2 + z^2 * x^2 - 4 * x * y * z = 0$ has a singular curve along x-axis, y-axis, z-axis and a triple point at the origin.





Lower Degree Spline Approximation of Rational Parametric Surfaces

For a rational parametric surface :

$$x(s,t) = \frac{X(s,t)}{W(s,t)}, y(s,t) = \frac{Y(s,t)}{W(s,t)}, z(s,t) = \frac{Z(s,t)}{W(s,t)}$$

Constructing lower degree rational spline approximations require solutions to sub-problems:

1. **Domain poles**
2. **Domain base points**
3. **Surface singularities**
4. **Complex parameter values**
5. **Infinite parameter values**

Triangulation and Display of Arbitrary Rational Parametric Surfaces, *Proceedings: IEEE Visualization '94 Conference*
Finite Representations of Real Parametric Curves and Surfaces, *Intl. Journal of Computational Geometry and Applications, (1995)*

Infinite parameter range

Consider the unit sphere:

implicit form: $f(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$

parametric form:

$$x = 2s/(1 + s^2 + t^2) \tag{20}$$
$$y = 2^t/(1 + s^2 + t^2) \tag{21}$$
$$z = 1 - s^2 - t^2/(1 + s^2 + t^2) \tag{22}$$

The point (0 , 0 , -1) can only be reached when both *s* and *t* tend to infinity.

Complex parameter values

We may need complex values to get real points

Consider the rational cubic curve:

implicit form: $f(x, y) = x^3 + x^2 + y^2 = 0$

parametric form: $x(s) = -s^2 + 1, y(s) = -s(s^2 + 1)$

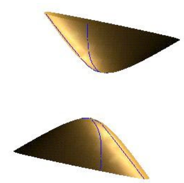
The origin can only be reached with $s = \sqrt{-1}$.

Poles

The denominator polynomial $f_4(s,t)$ may be 0, yielding a polynomial pole curve

Consider a hyperboloid of 2 sheets: implicit form:

$$f(x,y,z) = z^2 + yz + xz - y^2 - xy - x^2 - 1 = 0$$



parametric form:

$$x(s,t) = 4s/(5t^2 + 6st + 5s^2 - 1) \quad (23)$$
$$y(s,t) = 4t/(5t^2 + 6st + 5s^2 - 1) \quad (24)$$
$$z(s,t) = (5t^2 + 6st - 2t + 5s^2 - 2s + 1)/(5t^2 + 6st + 5s^2 - 1) \quad (25)$$

The problem arises from the polynomial pole curve $5t^2 + 6st - 2t + 5s^2 - 2s + 1 = 0$ in the parameter domain.

Base points

All the polynomials may equal 0 for some values of *s* and *t*, thus causing curves (seam curves) to be missing from the parametric surface

Hyperboloid of 1 sheet with seam curve gaps caused by two base points :

Handling Base points

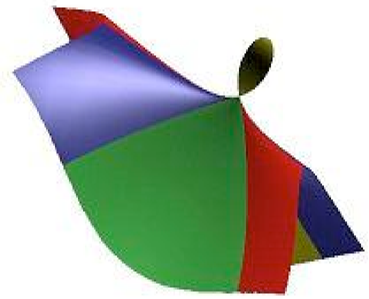
THEOREM : Let (a, b) be a base point of multiplicity q. Then for any $m \in R$, the image of a domain point approaching (a, b) along a line of slope m is given by $(X(m), Y(m), Z(m)W(m) =$

$$\sum_{i=0}^{q}(\frac{\partial^q X}{\partial s^{q-i}\partial t^i}(a,b))m^i \ldots \sum_{i=0}^{q}(\frac{\partial^q X}{\partial s^{q-i}\partial t^i}(a,b))m^i$$

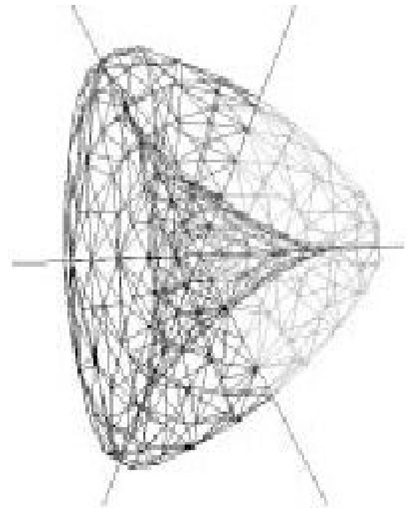
COROLLARY : If the curves $X(s,t) = 0, ..., W(s,t) = 0$ share t tangent lines at (a, b), then the seam curve $(X(m), Y(m), Z(m), W(m))$ has degree $q - t$. In particular, if $X(s,t) = 0$ have identical tangents at (a, b), then for all $m \in R$ the coordinates $(X(m), ..., W(m))$ represent a single point.

Parametric surfaces with a point, curve singularities

A Cubic Rational Surface:



The Steiner Rational Surface:



Algebraic Surface Blending, Joining, Least Squares Spline Approximations

Input: A collection of points, curves, derivative jets (scattered data) in 3D.

Output: A low degree, algebraic surface fit through the scattered set of points, curves, derivative jets, with prescribed higher order interpolation and least-squares approximation.

The mathematical model for this problem is a constrained minimization problem of the form :

$$\textit{minimize} \quad \mathbf{x}^T \mathbf{M_A}^T \mathbf{M_A} \mathbf{x} \quad \textit{subject to } \mathbf{M_I}\mathbf{x} = \mathbf{0},\ \mathbf{x}^T\mathbf{x} = 1,$$

$\mathbf{M_I}$ and $\mathbf{M_A}$ are interpolation and least-square approximation matrices, and $\mathbf{x}$ is a vector containing coefficients of an algebraic surface.

Theoretical Basis - I

Definition

Two algebraic surfaces $f(x,y,z) = 0$ and $g(x,y,z) = 0$ meet with C^k *rescaling continuity* at a point p or along an irreducible algebraic curve C if and only if there exists two polynomials $a(x,y,z)$ and $b(x,y,z)$, not identically zero at p or along C, such that all derivatives of $af - bg$ up to order k vanish at p or along C.

Theoretical Basis - II

Theorem

Let $g(x,y,z)$ and $h(x,y,z)$ be distinct, irreducible polynomials. If the surfaces $g(x,y,z) = 0$ and $h(x,y,z) = 0$ intersect transversally in a single irreducible curve C, then any algebraic surface $f(x,y,z) = 0$ that meets $g(x,y,z) = 0$ with C^k rescaling continuity along C must be of the form $f(x,y,z) = \alpha(x,y,z)g(x,y,z) + \beta(x,y,z)h^{k+1}(x,y,z)$. If $g(x,y,z) = 0$ and $h(x,y,z) = 0$ share no common components at infinity. Furthermore, the degree of $\alpha(x,y,z)g(x,y,z) \leq$ degree of $f(x,y,z)$ and the degree of $\beta(x,y,z)h^{k+1}(x,y,z) \leq$ degree of $f(x,y,z)$.

Higher-Order Interpolation and Least-Squares Approximation Using Implicit Algebraic Surfaces *ACM Transactions on Graphics, (1993)*

Quartic Joining Surfaces

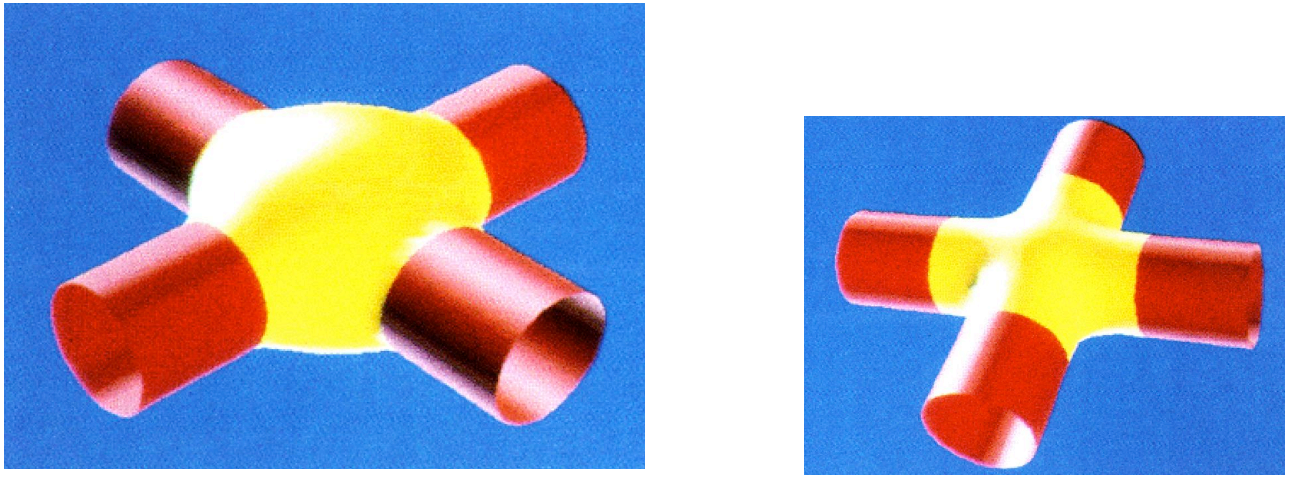


Figure: C^1 Interpolation at the Joins and Least-Squares Approximation in the Middle

Piecewise C^1 Cubic Fit

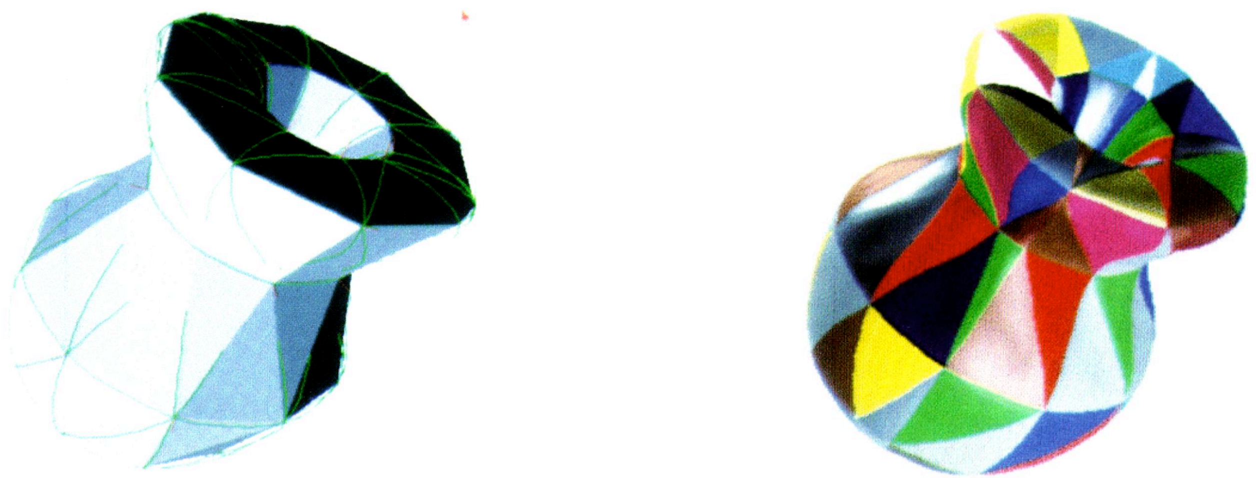
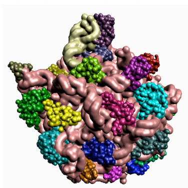
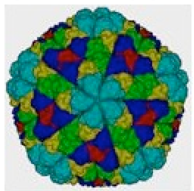
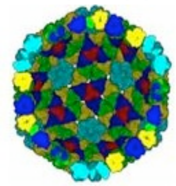
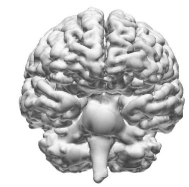
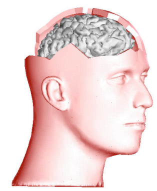
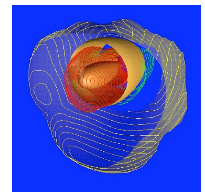


Figure: C^1 Cubic Rational Algebraic Spline

So what are Algebraic Splines, again ?

Collection (Complex) of smooth finite elements of polynomial (algebraic) curves and surfaces with prescribed order of continuity between the finite elements.

1. The splines are variously called Simplex, Box, Polyhedral depending on the support of the polynomial pieces.
2. The splines also can variously use the B-basis (B stands for Basis) or the BB-basis (BB stands for Bernstein-Bezier), or the C-basis (C for Chebyshev), etc. depending on the choice of polynomial basis
3. B-Splines (E.g. UBs or NUBs) or B-patches or Rational B-splines (e.g. NURBs) or T-Splines or X-splines etc. are just several examples of polynomial splines which are rational.

Additional Reading

- The references given below include the ones cited in the lecture slides. Please check for pdf's of these references on university computers from http://cvcweb.ices.utexas.edu/cvc/papers/papers.php

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The Mathematics of Surfaces III, edited by D. Handscomb, Oxford University Press, (1990), Chapter I, 3-48.
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Surface Fitting with Implicit Algebraic Surface Patches
Topics in Surface Modeling, edited by H. Hagen, SIAM Publications, (1992), Chapter 2, 23-52.

- S. Abhyankar, C.Bajaj "Parameterization of Algebraic Curves III"
http://www.cs.utexas.edu/~bajaj/cs384R08/reading/ParamIII.pdf