Lecture 11

Models: Algebraic Finite Elements
(Trimmed Blending/Lofted Surfaces)
Definition

Two algebraic surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$ meet with $C^k$ rescaling continuity at a point $p$ or along an irreducible algebraic curve $C$ if and only if there exists two polynomials $a(x, y, z)$ and $b(x, y, z)$, not identically zero at $p$ or along $C$, such that all derivatives of $af - bg$ up to order $k$ vanish at $p$ or along $C$. 

Theoretical Basis - I
Theoretical Basis - II

Theorem

Let $g(x, y, z)$ and $h(x, y, z)$ be distinct, irreducible polynomials. If the surfaces $g(x, y, z) = 0$ and $h(x, y, z) = 0$ intersect transversally in a single irreducible curve $C$, then any algebraic surface $f(x, y, z) = 0$ that meets $g(x, y, z) = 0$ with $C^k$ rescaling continuity along $C$ must be of the form $f(x, y, z) = \alpha(x, y, z)g(x, y, z) + \beta(x, y, z)h^{k+1}(x, y, z)$. If $g(x, y, z) = 0$ and $h(x, y, z) = 0$ share no common components at infinity. Furthermore, the degree of $\alpha(x, y, z)g(x, y, z) \leq$ degree of $f(x, y, z)$ and the degree of $\beta(x, y, z)h^{k+1}(x, y, z) \leq$ degree of $f(x, y, z)$. 

Algebraic Surface Blending, Joining, Least Squares Spline Approximations

**Input:** A collection of points, curves, derivative jets (scattered data) in 3D.

**Output:** A low degree, algebraic surface fit through the scattered set of points, curves, derivative jets, with prescribed higher order interpolation and least-squares approximation.

The mathematical model for this problem is a constrained minimization problem of the form:

$$\text{minimize } \mathbf{x}^T \mathbf{M}_A^T \mathbf{M}_A \mathbf{x} \quad \text{subject to } \mathbf{M}_I \mathbf{x} = \mathbf{0}, \quad \mathbf{x}^T \mathbf{x} = 1,$$

\(\mathbf{M}_I\) and \(\mathbf{M}_A\) are interpolation and least-square approximation matrices, and \(\mathbf{x}\) is a vector containing coefficients of an algebraic surface.
Quartic Joining Surfaces

Figure: $C^1$ Interpolation at the Joins and Least-Squares Approximation in the Middle
1. Triangulate each of the non-triangular polygonal faces of the given polyhedron $\mathcal{P}$. Any simple polygon is easily triangulable by adding non-intersecting inner diagonals.

2. Specify a unique “normal” vector at each vertex of $\mathcal{P}$. This provides a unique tangent plane for all patches which shall $C^1$ interpolate that vertex.

3. Next, construct a curvilinear wire frame by replacing each edge of $\mathcal{P}$ with a curve which $C^1$ interpolates the end points of the edge and the specified “normals”. Any remaining degrees of freedom of the $C^1$ interpolatory curve are used to select a desired shape of the curve and indirectly thereby a desired shape of the smoothing surface patch.

4. Specify normal vectors at each point along each of the edge curves. This provides the tangent planes for the two incident patches which shall $C^1$ interpolate the edge curves. If it is required that the individual patches are non-singular at the vertices of $\mathcal{P}$, then the variation of normals along different edge curves incident at the same vertex need also to be made compatible.

5. Finally, $C^1$ interpolate the three edge curves and curve normals of each face. The remaining degrees of freedom for each individual patch are chosen via weighted least squares to achieve a suitably shaped single-sheeted surface patch. The resulting surface patches yield a globally $C^1$ smooth curved model for the given polyhedron.
Smooth Watertight Surfaces I: A-patches
Smooth Watertight Surfaces II

As a result of least squares approximation of the function’s contour levels, we lead to the following computational model:

\[
\begin{align*}
\text{minimize} \quad & \| M_A x - b \|^2 \\
\text{subject to} \quad & M_1 x = z,
\end{align*}
\]

where \( M_I \in \mathbb{R}^{n_i \times 56} \) is a Hermite interpolation matrix, and \( M_A \in \mathbb{R}^{n_a \times 56} \) and \( b \in \mathbb{R}^{n_a} \) are matrix and vector, respectively, for contour level approximation, and \( x \in \mathbb{R}^{56} \) is a vector containing coefficients of a quintic algebraic surface \( f(x, y, z) = 0 \).

To find the nullspace of \( M_I \) in a computationally stable manner, the singular value decomposition (SVD) of \( M_I \) is computed \([82]\) where \( M_I \) is decomposed as \( M_I = U \Sigma V^T \) where \( U \in \mathbb{R}^{n_i \times n_i} \) and \( V \in \mathbb{R}^{56 \times 56} \) are orthonormal matrices, and \( \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_s) \in \mathbb{R}^{n_i \times 56} \) is a diagonal matrix with diagonal elements \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_s \geq 0 \) (\( s = \min\{n_i, 56\} \)). It is known that the rank \( r \) of \( M_I \) is the number of the positive diagonal elements of \( \Sigma \), and that the last \( 56 - r \) columns of \( V \) span the nullspace of \( M_I \). Hence, the nullspace of \( M_I \) is expressed as:

\[
\{ x \in \mathbb{R}^{56} \mid x = \sum_{j=1}^{56-r} w_j v_{j+r}, \text{where } w_i \in \mathbb{R}, \text{and } v_j \text{ is the } j\text{th column of } V \}, \text{ or } x = V_{56-r} w
\]

where \( V_{56-r} \in \mathbb{R}^{56 \times (56-r)} \) is made of the last \( 56-r \) columns of \( V \), and \( w \) a \((56-r)\)-vector. \( x = V_{56-r} w \) compactly expresses all the quintic surfaces which Hermite-interpolate the three quadric wires.

After substitution for \( x \), we lead to \( \| M_A x - b \| = \| M_A V_{56-r} w - b \| \). Then, an orthogonal matrix \( Q \in \mathbb{R}^{n_a \times n_a} \) is computed such that

\[
Q^T M_A V_{56-r} = R = \begin{pmatrix} R_1 \\ z \end{pmatrix}
\]

where \( R_1 \in \mathbb{R}^{(56-r) \times (56-r)} \) is upper triangular. (This factorization is called a \( Q-R \) factorization \([82]\)). Now, let

\[
Q^T b = \begin{pmatrix} c \\ d \end{pmatrix}
\]

where \( c \) is the first \( 56-r \) elements. Then, \( \| M_A V_{56-r} w - b \|^2 = \| Q^T M_A V_{56-r} w - Q^T b \|^2 = \| R_1 w - c \|^2 + \| d \|^2 \). The solution \( w \) can be computed by solving \( R_1 w = c \), from which the final fitting surface is obtained as \( x = V_{56-r} w \).
Surface Distance / Approximation

Let $q$ be the point on the surface which results in the distance. Then, the line in the direction of the normal of $f$ at $q$ must pass through $p$, and $q = p + t \frac{\nabla f(q)}{\|\nabla f(q)\|}$ where the absolute value of $t$ is the distance. From the Taylor's expansion,

$$0 = f(q) = f(p) + \nabla f(p) \cdot (t \frac{\nabla f(q)}{\|\nabla f(q)\|}) + \cdots .$$

Hence,

$$|t| \approx \frac{-f(p) \|\nabla f(q)\|}{\nabla f(p) \cdot \nabla f(q)} \quad (73)$$

is the first order approximation to the distance from $p$ to $f$. When $p$ is close to the surface, $\nabla f(p)$ is a good approximation to $\nabla f(q)$. In this case, the expression (73) becomes

$$|t| \approx \frac{-f(p) \|\nabla f(p)\|}{\nabla f(p) \cdot \nabla f(p)} = \frac{-f(p) \|\nabla f(p)\|}{\|\nabla f(p)\|^2} = \frac{f(p)}{\|\nabla f(p)\|^2} \overset{\text{def}}{=} \text{dist}_f(p).$$

This argument suggests that $\text{dist}_f(p)$, the weighted algebraic distance, be a good approximation to the geometric distance, and that

$$\sum_{\text{for all } p} \text{dist}_f(p)^2 = \sum_{\text{for all } p} \frac{f(p)^2}{\|\nabla f(p)\|^2} \quad (74)$$

be minimized instead of

$$\sum_{\text{for all } p} f(p)^2 \quad (75)$$

However, the solution which minimizes the expression (74) can not be easily expressed in closed form due to introduction of the weight $\|\nabla f(p)\|$.

This numerical intractability can be avoided by an iterative refinement algorithm. First, we compute $x_{(0)}$, coefficients of a surface $f_{(0)}$, such that (75), the sum of squares of algebraic distances, is minimized. To do this, $M_A = M_A_{(0)}$ is obtained as before. The gradient of $f_{(0)}$ gives an initial guess to $\nabla f(p)$. Then, dividing each row of $M_A$ by $\|\nabla f_{(0)}(p)\|$ for each corresponding $p$ results in $M_A(1)$ which is, then, singular-value-decomposed to compute $x_{(1)}$ and $f_{(1)}$. This process is repeated further producing a sequence of $f_{(k)}$ which refines the solution. In each iteration, $f_{(k)}$ is expected to be a better approximation to the surface we are trying to find.
Heart Model via X-section Contour Lofting

First segment the heart into four independent planar contour stacks from MRI data: background (0), heart muscle (81), left ventricle (162), right ventricle (243) and then loft (skin) the planar contour stacks.

Simulation of the electronic activity of the heart.
Lofted Surfaces
Cross-Sectional Data Fitting
Lofted Surfaces
Patient-specific vascular models

(a) Volume rendering

(b) Smooth Surf. reconstruction

Abdominal Aorta

(c) Skeleton

(d) Control Hex mesh

(e) Solid NURBS
Additional Reading

• The references given below include the ones cited in the lecture slides. Please check for pdf’s of these additional references on university computers from http://cvcweb.ices.utexas.edu/cvc/papers/papers.php

  • C.Bajaj and A. Gillette  **Tutorial Notes on “Algebraic Finite Elements with Applications”,** Chap 4, 2010.

  • C.Bajaj  **Tutorial Notes on “Computational Structural Bioinformatics I: Spatially Realistic Models from Imaging”,** Chap 2, 2010.

  • C.Bajaj  **Tutorial Notes on “Computational Structural Bioinformatics II: Molecular Models”,** Chap 2, 2010.

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