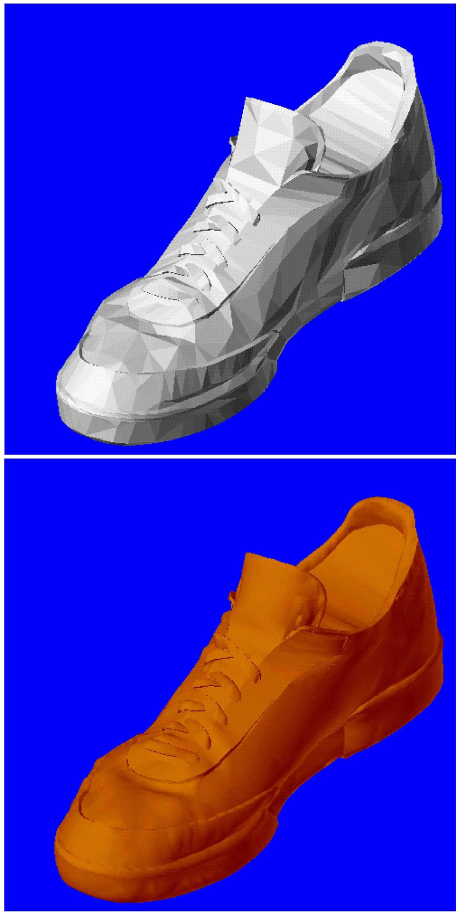
Curves, Surfaces and Segments, Patches



- Conics: Curves and Quadrics: Surfaces
  - Implicit form
  - Parametric form
- Rational Bézier Forms and Join Continuity
- Recursive Subdivision of Bézier Curve segments
- Recursive Subdivision of Bézier Surface patches

Conic Curves

Conic Sections (Implicit form)

- Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a, b > 0$$

- Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad a, b > 0$$

- Parabola

$$y^2 = 4ax \quad a > 0$$

Conic Sections (Parametric form)

- Ellipse

$$
\begin{aligned}
x(t) &= a\frac{1-t^2}{1+t^2} \\
y(t) &= b\frac{2t}{1+t^2} \quad (-\infty < t < +\infty)
\end{aligned}
$$

- Hyperbola

$$
\begin{aligned}
x(t) &= a\frac{1+t^2}{1-t^2} \\
y(t) &= b\frac{2t}{1-t^2} \quad (-\infty < t < +\infty)
\end{aligned}
$$

- Parabola

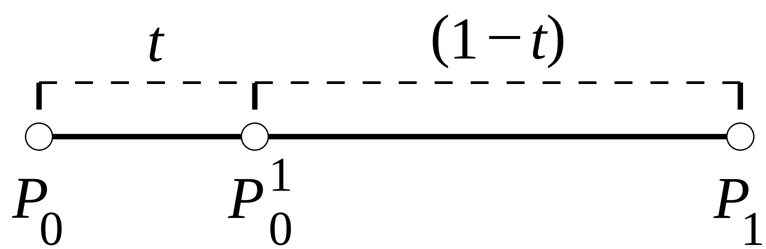
$$
\begin{aligned}
x(t) &= at^2 \\
y(t) &= 2at \;\; (-\infty < t < +\infty)
\end{aligned}
$$

Constructing Curve Segments

Linear blend:

- Line segment from an affine combination of points

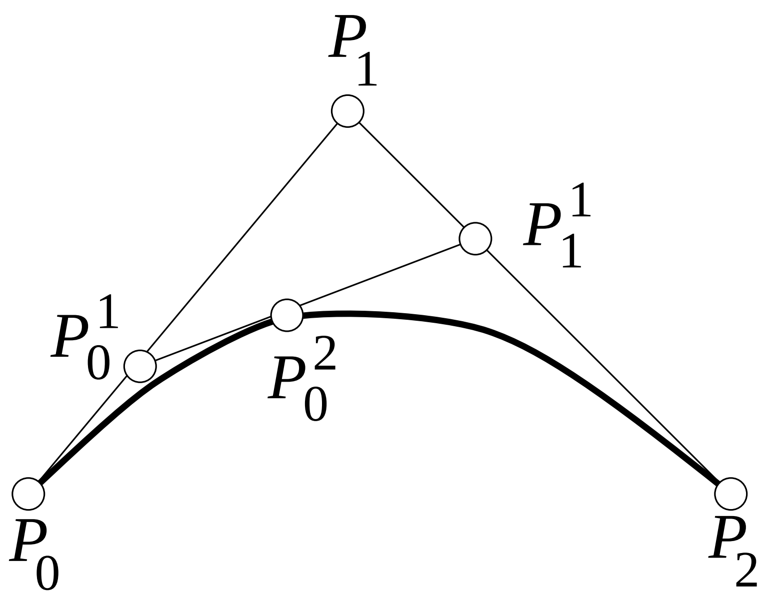
$$P_0^1(t) = (1-t)P_0 + tP_1$$



Quadratic blend:

- Quadratic segment from an affine combination of line segments

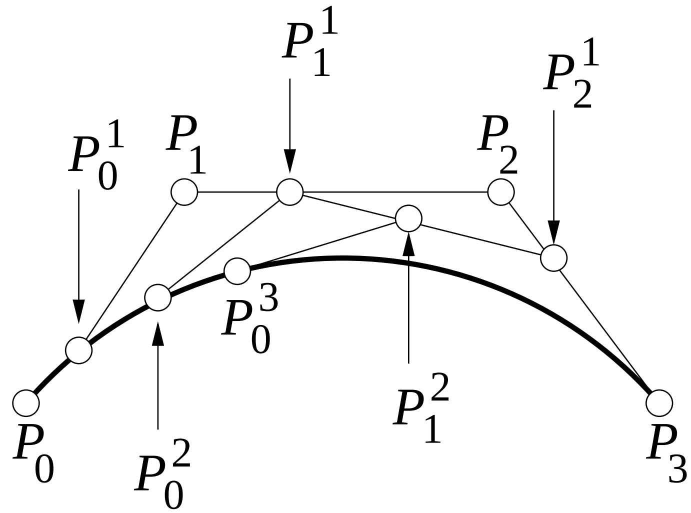
$$
\begin{aligned}
P_0^1(t) &= (1-t)P_0 + tP_1 \\
P_1^1(t) &= (1-t)P_1 + tP_2 \\
P_0^2(t) &= (1-t)P_0^1(t) + tP_1^1(t)
\end{aligned}
$$



Cubic blend:

- Cubic segment from an affine combination of quadratic segments

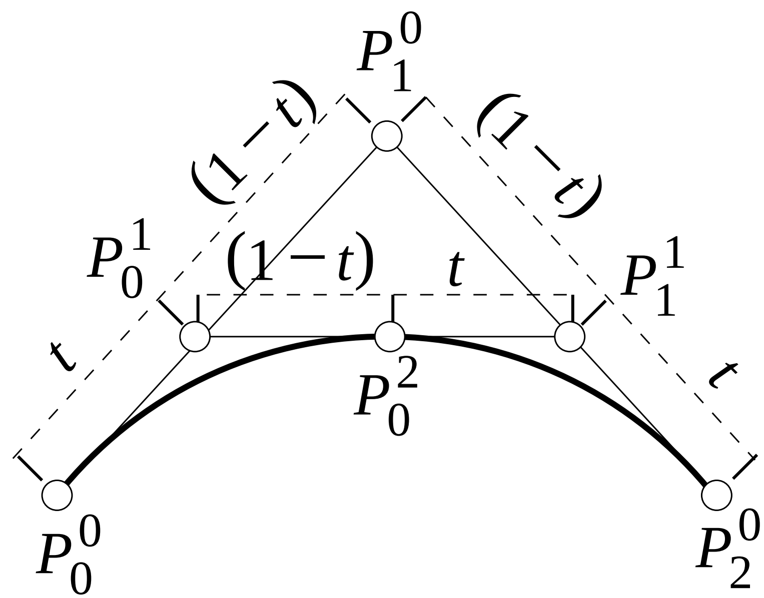
$$
\begin{aligned}
P_0^1(t) &= (1-t)P_0 + tP_1 \\
P_1^1(t) &= (1-t)P_1 + tP_2 \\
P_0^2(t) &= (1-t)P_0^1(t) + tP_1^1(t) \\
P_2^1(t) &= (1-t)P_2 + tP_3 \\
P_1^2(t) &= (1-t)P_1^1(t) + tP_2^1(t) \\
P_0^3(t) &= (1-t)P_0^2(t) + tP_1^2(t)
\end{aligned}
$$



- The pattern should be evident for higher degrees

Geometric view (de Casteljau Algorithm):

- Join the points P_i by line segments
- Join the $t : (1-t)$ points of those line segments by line segments
- Repeat as necessary
- The $t : (1-t)$ point on the final line segment is a point on the curve
- The final line segment is tangent to the curve at t



Expanding Terms (Basis Polynomials):

- The original points appear as coefficients of *Bernstein polynomials*

$$
\begin{aligned}
P_0^0(t) &= P_0 1 \\
P_0^1(t) &= (1-t)P_0 + tP_1 \\
P_0^2(t) &= (1-t)^2 P_0 + 2(1-t)tP_1 + t^2 P_2 \\
P_0^3(t) &= (1-t)^3 P_0 + 3(1-t)^2 tP_1 + 3(1-t)t^2 P_2 + t^3 P_3 \\
P_0^n(t) &= \sum_{i=0}^{n} P_i B_i^n(t)
\end{aligned}
$$

where $B_i^n(t) = \frac{n!}{(n-i)!i!}(1-t)^{n-i}t^i = \left(\begin{array}{c} n \\ i \end{array} \right) (1-t)^{n-i}t^i$

- The Bernstein polynomials of degree n form a basis for the space of all degree-n polynomials

Recursive evaluation schemes:

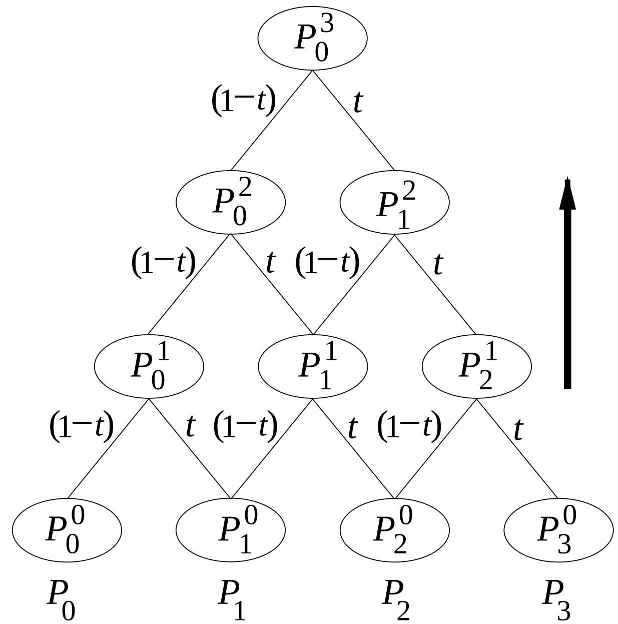
- To obtain curve points (upward diagram):
  - Start with given points and form successive, pairwise, affine combinations

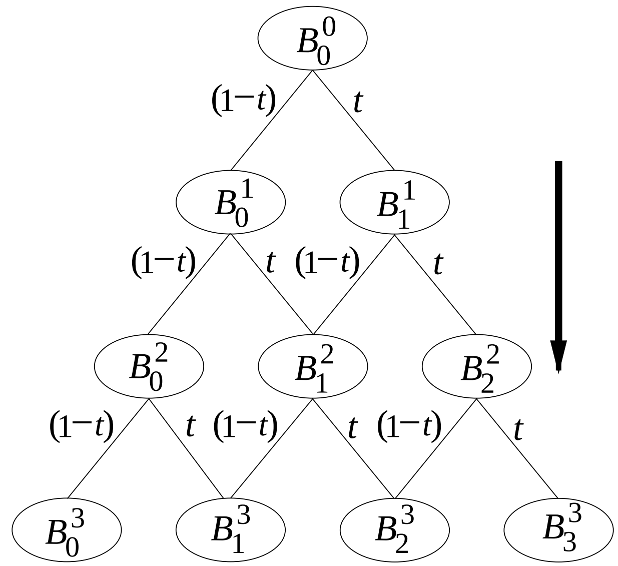
$$
\begin{aligned}
P_i^0 &= P_i \\
P_i^j &= (1-t)P_i^{j-1} + tP_{i+1}^{j-1}
\end{aligned}
$$

  - The generated points P_i^j are the *deCasteljau points*
- To obtain basis polynomials (downward diagram):
  - Start with 1 and form successive, pairwise, affine combinations

$$
\begin{aligned}
B_0^0 &= 1 \\
B_i^j &= (1-t)B_i^{j-1} + tB_{i+1}^{j-1}
\end{aligned}
$$

  where $B_r^s = 0$ when $r < 0$ or $r > s$


P_0^3
(1−t)
t
P_0^2
P_1^2
(1−t)
t
(1−t)
t
P_0^1
P_1^1
P_2^1
(1−t)
t
(1−t)
t
(1−t)
t
P_0^0
P_1^0
P_2^0
P_3^0
P_0
P_1
P_2
P_3



B_0^0
(1−t)
t
B_0^1
B_1^1
(1−t)
t
(1−t)
t
B_0^2
B_1^2
B_2^2
(1−t)
t
(1−t)
t
(1−t)
t
B_0^3
B_1^3
B_2^3
B_3^3

Bernstein-Bézier (BB) Splines

Bernstein-Bézier (BB) Curve Segments and their Properties

Definition:

- A degree n (order $n+1$) *Bernstein-Bézier curve segment* is

$$P(t) = \sum_{i=0}^{n} \vec{P}_i B_i^n(t)$$

where the $\vec{P}_i$ are k-dimensional *control points*.

Rational Quadratic BB Forms

Quadratic Rational BB Form:

- Homogeneous form

$$\begin{bmatrix} x(t) \\ y(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ w_0 \end{bmatrix} B_0^2(t) + \begin{bmatrix} x_1 \\ y_1 \\ w_1 \end{bmatrix} B_1^2(t) + \begin{bmatrix} x_2 \\ y_2 \\ w_2 \end{bmatrix} B_2^2(t)$$

$$= \begin{bmatrix} x_0 B_0^2(t) + x_1 B_1^2(t) + x_2 B_2^2(t) \\ y_0 B_0^2(t) + y_1 B_1^2(t) + y_2 B_2^2(t) \\ w_0 B_0^2(t) + w_1 B_1^2(t) + w_2 B_2^2(t) \end{bmatrix}$$

- Rational (projected) form

$$\begin{bmatrix} \bar{x}(t) \\ \bar{y}(t) \end{bmatrix} = \begin{bmatrix} \frac{x_0 B_0^2(t)+x_1 B_1^2(t)+x_2 B_2^2(t)}{w_0 B_0^2(t)+w_1 B_1^2(t)+w_2 B_2^2(t)} \\ \frac{y_0 B_0^2(t)+y_1 B_1^2(t)+y_2 B_2^2(t)}{w_0 B_0^2(t)+w_1 B_1^2(t)+w_2 B_2^2(t)} \end{bmatrix}$$

$$= \frac{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} B_0^2(t) + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} B_1^2(t) + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} B_2^2(t)}{w_0 B_0^2(t) + w_1 B_1^2(t) + w_2 B_2^2(t)}$$

Conversions:

- Conic parameterization elements in BB form

$$
\begin{aligned}
2t &= B_1^2(t) + 2B_2^2(t) \\
1 - t^2 &= B_0^2(t) + B_1^2(t) \\
1 + t^2 &= B_0^2(t) + B_1^2(t) + 2B_2^2(t) \\
t^2 &= B_2^2(t)
\end{aligned}
$$

Conics as Rational Bézier Curves

Conics as NURBS (Ellipse)

- Rational Bézier

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{\begin{bmatrix} a(1-t^2) \\ b(2t) \end{bmatrix}}{1+t^2}$$

$$= \frac{\begin{bmatrix} aB_0^2(t) + aB_1^2(t) + 0B_2^2(t) \\ 0B_0^2(t) + bB_1^2(t) + b2B_2^2(t) \end{bmatrix}}{B_0^2(t) + B_1^2(t) + 2B_2^2(t)}$$

$$= \frac{\begin{bmatrix} a \\ 0 \end{bmatrix} B_0^2(t) \begin{bmatrix} a \\ b \end{bmatrix} B_1^2(t) \begin{bmatrix} 0 \\ 2b \end{bmatrix} B_2^2(t)}{B_0^2(t) + B_1^2(t) + 2B_2^2(t)}$$

which implies

$$\begin{array}{lll} w_0 = 1 & x_0 = a & y_0 = 0 \\ w_1 = 1 & x_1 = a & y_1 = b \\ w_2 = 2 & x_2 = 0 & y_2 = 2b \end{array}$$

Conics as NURBS (Hyperbola)

- Rational Bézier

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{\begin{bmatrix} a(1+t^2) \\ b(2t) \end{bmatrix}}{1-t^2}$$

$$= \frac{\begin{bmatrix} aB_0^2(t) + aB_1^2(t) + a2B_2^2(t) \\ 0B_0^2(t) + bB_1^2(t) + b2B_2^2(t) \end{bmatrix}}{B_0^2(t) + B_1^2(t)}$$

$$= \frac{\begin{bmatrix} a \\ 0 \end{bmatrix} B_0^2(t) \begin{bmatrix} a \\ b \end{bmatrix} B_1^2(t) \begin{bmatrix} 2a \\ 2b \end{bmatrix} B_2^2(t)}{B_0^2(t) + B_1^2(t) + 0B_2^2(t)}$$

which implies

$$\begin{array}{lll} w_0 = 1 & x_0 = a & y_0 = 0 \\ w_1 = 1 & x_1 = a & y_1 = b \\ w_2 = 0 & x_2 = 2a & y_2 = 2b \end{array}$$

Conics as NURBS (Parabola)

- Rational Bézier

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{\begin{bmatrix} a(t^2) \\ a(2t) \end{bmatrix}}{1}
$$

$$
= \frac{\begin{bmatrix} 0B_0^2(t) + 0B_1^2(t) + aB_2^2(t) \\ 0B_0^2(t) + aB_1^2(t) + a2B_2^2(t) \end{bmatrix}}{B_0^2(t) + B_1^2(t) + B_2^2(t)}
$$

$$
= \frac{\begin{bmatrix} 0 \\ 0 \end{bmatrix} B_0^2(t) \begin{bmatrix} 0 \\ a \end{bmatrix} B_1^2(t) \begin{bmatrix} a \\ 2a \end{bmatrix} B_2^2(t)}{B_0^2(t) + B_1^2(t) + B_2^2(t)}
$$

which implies

$$
\begin{array}{lll} w_0 = 1 & x_0 = 0 & y_0 = 0 \\ w_1 = 1 & x_1 = 0 & y_1 = a \\ w_2 = 1 & x_2 = a & y_2 = 2a \end{array}
$$

Not Unique

- x, y, w are not unique
  - Numerator and denominator can be multiplied by a common (positive) factor
- The following example is a common alternative form:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} w_0 B_0^2(t) + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} w_1 B_1^2(t) + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} w_2 B_2^2(t)}{w_0 B_0^2(t) + w_1 B_1^2(t) + w_2 B_2^2(t)}$$

which derives from rewriting

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \longrightarrow w \begin{bmatrix} \bar{x} \\ \bar{y} \\ 1 \end{bmatrix}$$

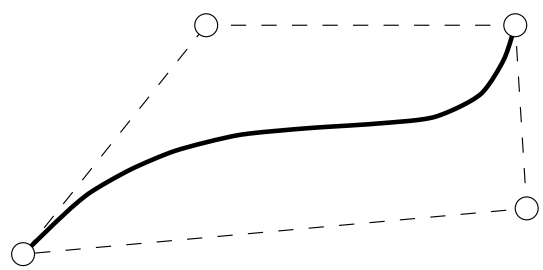
Bernstein-Bézier Curve Properties

Convex Hull:

$\sum_{i=0}^{n} B_i^n(t) = 1,\ B_i^n(t) \geq 0$ for $t \in [0,1]$

$\Longrightarrow P(t)$ is a convex combination of the $\vec{P}_i$ for $t \in [0,1]$

$\Longrightarrow P(t)$ lies within convex hull of $\vec{P}_i$ for $t \in [0,1]$



Affine Invariance:

- A Bernstein-Bézier curve is an affine combination of its control points
- Any affine transformation of a curve is the curve of the transformed control points

$$T\left(\sum_{i=0}^{n} P_i B_i^n(t)\right) = \sum_{i=0}^{n} T(P_i) B_i^n(t)$$

- *This property does not hold for projective transformations!*

Interpolation:

$$B_0^n(0) = 1, B_n^n(1) = 1, \sum_{i=0}^{n} B_i^n(t) = 1, B_i^n(t) \geq 0 \text{ for } t \in [0,1]$$

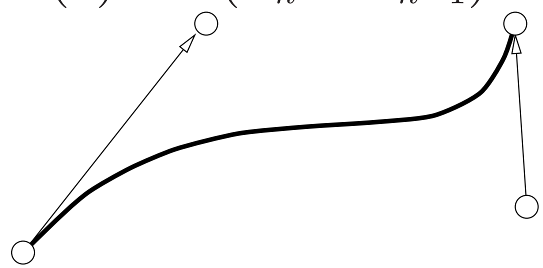
$$\Longrightarrow B_i^n(0) = 0 \text{ if } i \neq 0, B_i^n(1) = 0 \text{ if } i \neq n$$

$$\Longrightarrow P(0) = P_0, P(1) = P_n$$

Derivatives:

$$\frac{d}{dt}B_i^n(t) = n\left(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right)$$

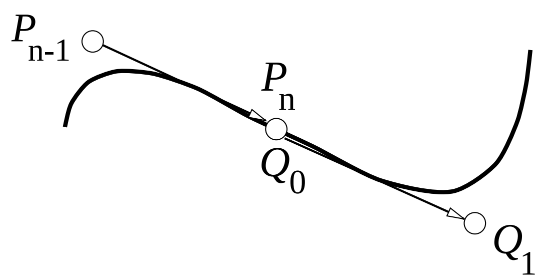
$$\Longrightarrow P'(0) = n(\vec{P}_1 - \vec{P}_0), P'(1) = n(\vec{P}_n - \vec{P}_{n-1})$$



Smoothly Joined Curve Segments (G^1 continuity)

- Let P_{n-1}, P_n be the last two control points of one segment
- Let Q_0, Q_1 be the first two control points of the next segment

$$\begin{aligned} P_n &= Q_0 \\ (P_n - P_{n-1}) &= \beta(Q_1 - Q_0) \text{ for some } \beta > 0 \end{aligned}$$



Recurrence, Subdivision:

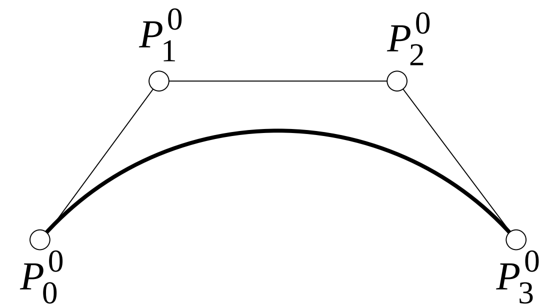
$$B_i^n(t) = (1-t)B_i^{n-1} + tB_{i-1}^{n-1}(t)$$

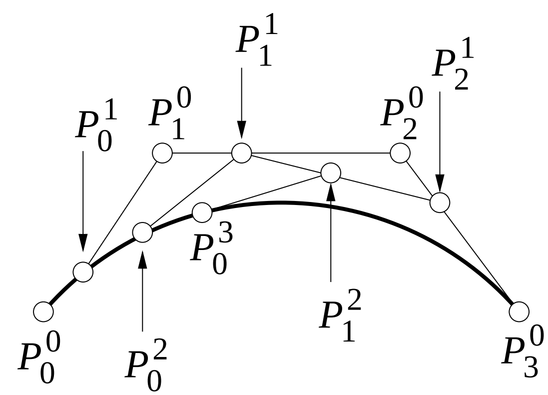
$\Longrightarrow$ deCasteljau's algorithm:

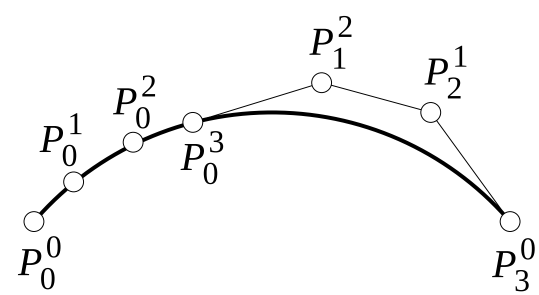
$$\begin{aligned} P(t) &= P_o^n(t) \\ P_i^k(t) &= (1-t)P_i^{k-1}(t) + t\ P_{i+1}^{k-1}(t) \\ P_i^0 &= P_i \end{aligned}$$

Use to evaluate point at t, or subdivide into two new curves:

- $P_0^0, P_0^1, \ldots P_0^n$ are the control points for the left half
- $P_n^0, P_{n-1}^1, \ldots P_0^n$ are the control points for the right half


P_1^0
P_2^0
P_0^0
P_3^0


P_1^1
P_2^1
P_0^1
P_1^0
P_2^0
P_0^3
P_1^2
P_0^0
P_0^2
P_3^0


P_1^2
P_2^1
P_0^2
P_0^1
P_0^3
P_0^0
P_3^0

Quadric Surfaces

Implicit form

- Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad a, b, c > 0$$

- Hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad a, b, c > 0$$

- Hyperbolic Paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad a, b, c > 0$$

- Parabolic

$$y^2 = 4ax \quad a > 0$$

Parametric form

- Ellipsoid

$$
\begin{aligned}
x(s,t) &= a\frac{1-s^2-t^2}{1+s^2+t^2} \\
y(s,t) &= b\frac{2s}{1+s^2+t^2} \quad (-\infty < s < +\infty) \\
z(s,t) &= b\frac{2t}{1+s^2+t^2} \quad (-\infty < t < +\infty)
\end{aligned}
$$

- Paraboloid

$$
\begin{aligned}
x(s,t) &= a(s^2+t^2) \\
y(s,t) &= as \quad (-\infty < s < +\infty) \\
z(s,t) &= at \quad (-\infty < t < +\infty)
\end{aligned}
$$

Tensor Product Bernstein-Bézier Patches

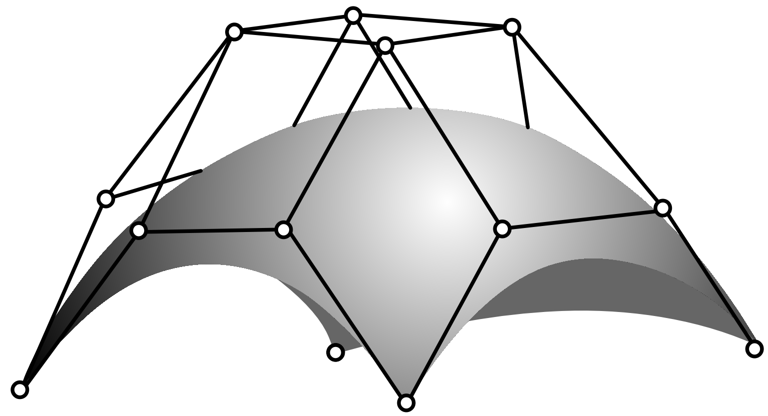
Tensor Patches:

- The *control polygon* is the polygonal mesh with vertices $P_{i,j}$
- The *patch basis functions* are products of Bézier curve basis functions

$$P(s,t) = \sum_{i=0}^{n} \sum_{j=0}^{n} P_{i,j} B_{i,j}^{n}(s,t)$$

where

$$B_{i,j}^{n}(s,t) = B_{i}^{n}(s) B_{j}^{n}(t)$$



Properties:

- Tensor Bernstein-Bézier Patch basis functions *sume to one*

$$\sum_{i=0}^{n}\sum_{j=0}^{n} B_i^n(s)B_j^n(t) = 1$$

- Patch basis functions are *nonnegative* on [0, 1] $\times$ [0, 1]

$$B_i^n(s)B_j^n(t) \geq 0 \text{ for } 0 \leq s, t \leq 1$$

$\Longrightarrow$ Surface patch is in the *convex hull* of the control points
$\Longrightarrow$ Surface patch is *affinely invariant*
(Transform the patch by transforming the control points)

Subdivision, Recursion, Evaluation:

- As for curves in each variable separately and independently
- Normals must be computed from partial derivatives

Partial Derivatives:

- Ordinary derivative in each variable separately':

$$\frac{\partial}{\partial s}P(s,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} P_{i,j}\left[\frac{d}{ds}B_i^n(s)\right]B_j^n(t)$$

$$\frac{\partial}{\partial t}P(s,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} P_{i,j}B_i^n(s)\left[\frac{d}{dt}B_j^n(t)\right]$$

- Each of the above is a *tangent vector* in a parametric direction
- Surface is *regular* at each (s,t) where these two vectors are linearly independent
- The (unnormalized) *surface normal* is given at any regular point by

$$\pm\left[\frac{\partial}{\partial s}P(s,t)\times\frac{\partial}{\partial t}P(s,t)\right]$$

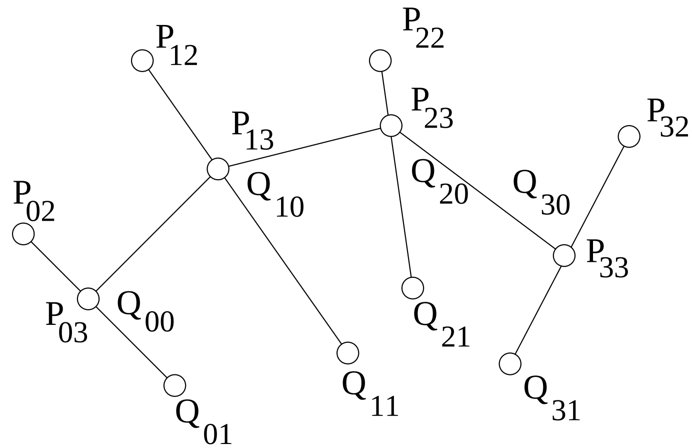
(the sign dictates what is the *outward pointing normal*)

- In particular, the *cross-boundary tangent* is given by (e.g., for the $s = 0$ boundary):

$$n \sum_{i=0}^{n} \sum_{j=0}^{n} (P_{1,j} - P_{0,j}) B_j^n(t)$$

(and similarly for the other boundaries)

Smoothly Joined Tensor Bernstein-Bézier Patches:



- Can be achieved by ensuring that

$$(P_{i,n} - P_{i,n-1}) = \beta(Q_{i,1} - Qi, 0) \text{ for } \beta > 0$$

(and correspondingly for other boundaries)

Rendering via Subdivision:

- Divide up into polygons:
  1. By stepping

$$
\begin{aligned}
s &= 0, \delta, 2\delta, \ldots, 1 \\
t &= 1, \gamma, 2\gamma, \ldots, 1
\end{aligned}
$$

     and joining up sides and diagonals to produce a triangular mesh
  2. By subdividing and rendering the control polygon

Barycentric Triangular Bernstein-Bézier Patches

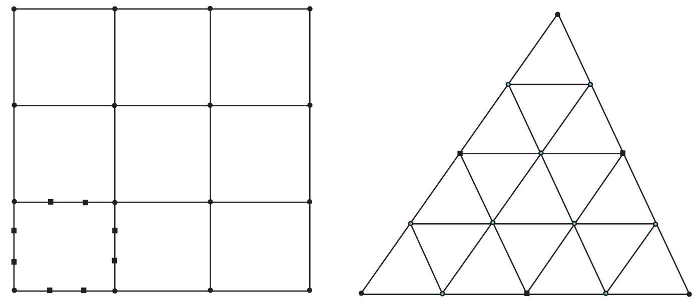
de Casteljau Revisited Barycentrically:

- Linear blend expressed in barycentric terms

$$(1-t)P_0 + tP_1 = rP_0 + tP_1 \quad \text{where} \quad r + t = 1$$

- Higher powers and a symmetric form of the Bernstein polynomials:

$$\begin{aligned} P(t) &= \sum_{i=0}^{n} P_i \left(\frac{n!}{i!(n-i)!} \right) (1-t)^{n-i} t^i \\ &= \sum_{\substack{i+j=n \\ i \geq 0, j \geq 0}} P_i \left(\frac{n!}{i!j!} \right) t^i r^j \quad \text{where} \quad r + t = 1 \\ &\Longrightarrow \sum_{\substack{i+j=n \\ i \geq 0, j \geq 0}} P_{ij} B_{ij}^n (r, t) \end{aligned}$$



- Examples

$$\{B^0_{00}(r,t)\} = \{1\}$$
$$\{B^1_{01}(r,t), B^1_{10}(r,t)\} = \{r, t\}$$
$$\{B^2_{02}(r,t), B^2_{11}(r,t), B^2_{20}(r,t)\} = \{r^2, 2rt, t^2\}$$
$$\{B^3_{03}(r,t), B^3_{12}(r,t), B^3_{21}(r,t), B^3_{30}(r,t)\} = \{r^3, 3r^2t, 3rt^2, t^3\}$$

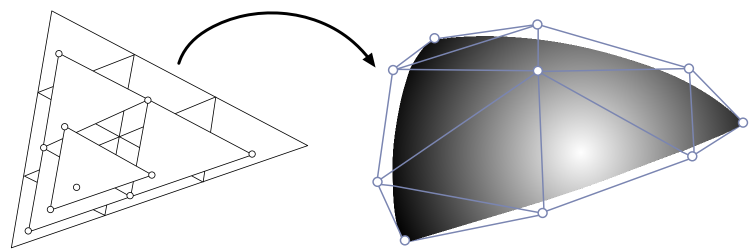
Surfaces – Barycentric Blends on Triangles:

- Formulas

$$P(r,s,t) = \sum_{\substack{i+j+k=n \\ i\geq 0, j\geq 0, k\geq 0}} P_{ijk} B^n_{ijk}(r,s,t)$$

$$B^n_{ijk}(r,s,t) = \frac{n!}{i!j!k!} r^i s^j t^k$$

Triangular Bézier Surface Patches

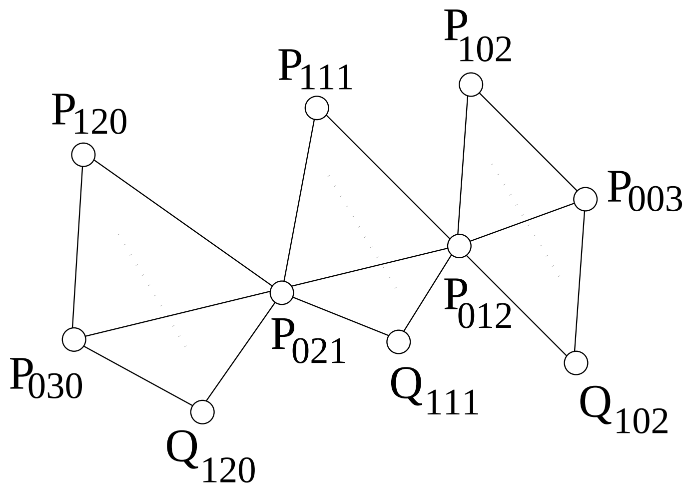


Triangular deCasteljau:

- Join adjacently indexed P_{ijk} by triangles
- Find $r : s : t$ barycentric point in each triangle
- Join adjacent points by triangles
- Repeat
  - Final point is the surface point $P(r, s, t)$
  - final triangle is tangent to the surface at $P(r, s, t)$
- Triangle up/down schemes become tetrahedral up/down schemes

Properties:

- Each boundary curve is a Bézier curve
- Patches will be joined smoothly if pairs of boundary triangles are planar as shown



Reading Assignment and News

Before the next class please review Chapter 10 and its practice exercises, of the recommended text.

(Recommended Text: Interactive Computer Graphics, by Edward Angel, Dave Shreiner, 6th edition, Addison-Wesley)

Please track Blackboard for the most recent Announcements and Project postings related to this course.

(http://www.cs.utexas.edu/users/bajaj/graphics2012/cs354/)