Question 1. Give one similarity and one difference between recursive subdivision schemes and fractals in the way they generate objects.

Answer 1. Subdivision schemes and fractals are similar because they both involve refining an object to a certain recursive depth. For subdivision, this can be expressed as the subdivision depth. For fractals specified as L-systems, this can be expressed as the L-system recursion depth. In addition, both subdivision and fractals involve replacing existing edges or vertices with new edges or more vertices in order to generate the next iteration of the refinement.

Subdivision and fractals are different because fractals can usually be refined using only local information. For instance, one edge of the initial triangle for the Koch snowflake can be refined to arbitrary depth without any knowledge of the location of the other two edges. On the other hand, many subdivision schemes, such as the Four Point Scheme, need to know a substantial neighborhood of points in iteration \( j \) in order to generate an iteration \( j+1 \) point, so local knowledge is not sufficient. Another difference is that many fractals are interpolatory, meaning that the fractal will always pass through any vertex generated in a prior iteration (like the Koch snowflake or Sierpinski triangle). Many forms of subdivision, on the other hand, are approximatory, meaning that the control points in iteration \( j \) do not necessarily carry over to the control points in iteration \( j+1 \) (such as in Catmull-Clark refinement).

Question 2. What is the limit curve when the Four Point Scheme is applied to a square? Assume that vertices are spaced out equally on the perimeter of the square and that there are at least 3 vertices on every side of the square (i.e. there can be internal vertices within a side). What is the limit surface when the Catmull-Clark Subdivision Scheme is applied to a cube? What can you say happens to sharp corners in general when these two subdivision schemes are applied?

Answer 2. When the Four Point Scheme is applied to a square with at least three vertices on each side, the limit curve (the curve after an infinite number of subdivisions) will be a square with rounded corners. Because the Four Point Scheme is interpolatory, the curve must pass through all of the iteration 0 control points, which prevents it from becoming a circle (since there was at least one control point on each edge of the original square). Subdivision does round out the corners, although these rounded corners still pass through the control points at the corners of the original square.

When Catmull-Clark is applied to a cube that is specified only at the corners of the cube, the limit surface will be roughly a sphere. If Catmull-Clark is applied to a cube with internal vertices on its faces and edges, however, then the limit surface looks like a cube with smoothed corners. Just like the earlier case of the square, the internal control points constrain the subdivision so that the refined surface passes almost through the original control points (almost through the original control points because Catmull-Clark is approximatory).

In general, subdivision schemes smooth out sharp corners in the objects on which they are applied.
Question 3. Describe how you would design a wide range of smooth 3-D models with holes if all you have are a 2-D sketching system for line segments (like the one from this project), a surface subdivider, and the ability to create surfaces of rotation from 2-D models. Provide pseudocode.

Answer 3. Here are two ways to create smooth 3-D models with holes using the provided tools.

**Approach 1:** This method produces a smooth 3-D model that has a hole that extends all the way through it, such as the hole in an apple that has had its core removed as well as the stem and the center of the base (so that you could see through the apple if you were looking at it from above). In this case, the hole is visible from the surface of the object and lies on the axis of symmetry.

Consider the cross section of the desired 3-D model where the axis of symmetry runs along the cross section. Align the cross section so the axis of symmetry is running up and down in the y direction. Draw the cross section using one or more closed loops, and then take only the right or left half of the cross section. In the case of a sphere with a penetrating hole, the cross section will be a pair of congruent semi-circles that are not touching. The algorithm considers either the left or right semi-circle, which will be used in the rotation step.

Pick the axis of rotation so that it goes right through the center of the location of the hole, which should be lying at some horizontal offset between the halves of the original cross section. When the curve is rotated about this axis, some empty space will be included in the center of the model. This bit of space, after the entire rotation, will form the eventual hole going through the object.

Now take either the left half or the right half of the 2-D cross section and rotate it completely around the axis of rotation, forming a 3-D model with a hole in it. Use the surface subdivider to smooth out the model, producing a new smooth model.

Since there are few restrictions on the 2-D curve before rotating it to generate a 3-D model, we can use a variety of 2-D curves to generate a variety of 3-D models. In this approach, we can generate cylinders, cones, spheres, and other objects, all with a penetrating hole. We can also generate a torus starting with an initial cross section of a circle, where the donut hole in the torus is the penetrating hole.

**Approach 2:** This method produces a smooth 3-D model that has a hole inside of the volume, although the hole is not visible from the surface. An example of such a model would be a peach where the pit is magically removed without disturbing any of the fruit flesh. In this case, the hole lies on the axis of symmetry and is not visible from the surface of the object. Although the hole will not be visible in a surface renderer, such models are still used in programs that can view arbitrary cross sections of volumetric data.

Consider the cross section of the desired 3-D model where the axis of symmetry runs along the cross section. The cross section should include the surface of the object as well as the bounds of all of its internal holes. There will be one outer curve for the surface and n inner curves inside of it, where n is the number of holes. In the case of a sphere with one spherical hole at its center, the cross section is two concentric circles.

Align the cross section so that the axis of symmetry is running up and down in the y direction. Place the axis of rotation along the axis of symmetry and take either the right half or left half of the cross section. Rotate this half about the axis of rotation and generate the 3-D model. Use the surface subdivider to smooth out the model, producing a new smooth model.

Since there are few restrictions on the 2-D curve before rotating it to generate a 3-D model, we can use a variety of 2-D curves to generate a variety of 3-D models. In this approach, the outer surface of the final model can be a cylinder, cone, sphere, or other object.