Queensland University of Technology

SCHOOL OF MECHANICAL, MANUFACTURING & MEDICAL ENGINEERING

MEN170: SYSTEMS MODELLING AND SIMULATION

6: PROBABILISTIC MODELS : MARKOV CHAINS.

6.1 Introduction: Suppose we observe some characteristic of a system such as level of water in a dam, the temperature of the oven (both continuously changing), state of a machine (discrete changes only), at some discrete points in time (0,1,2,3..) and let X_t be the value of this characteristic at time t. In most situations, X_t will not be known with certainty until the time arrives and may be viewed as a random variable.In general, X_t depends on all the previous stages X_0, X_1, ... X_{t-1}. **Discrete time stochastic process** is simply a description of the relation between these random variables X_0, X_1, X_2 ... X_t. One set of characteristics of a system is the states of the system. If we know all the possible states of the system, then the behaviour of the system is completely described by its states. A system may have finite or infinite number of states. Here, we are concerned with only finite states systems. If the state of a system can change in some probabilistic fashion at fixed or random interval in time, we have a **stochastic process.** An important feature of a stochastic process is the way in which the past behaviour influences the future behaviour. Suppose **X**(t) describes the state of the system and has n values. That is, at a given time, $X_1(t)$, $X_2(t)$ ...$X_n(t)$ are the possible states of the system. $X_i(t)$ could be demand of a product or number of parts waiting to be processed in a machine or price of a share or condition of a machine etc. The system will move from one state to another with some random fashion. That is, there is a probability attached to this. For example, the condition of the machine will change from working to failed with a probability of 0.12 and is called the **transition probability**. This value may be constant or may change with time. For example, we know that old machines fail quite frequently than new machines. Then the probability of changing from working to fail state of the machine will increase with time. Let us suppose that **p**(t) represents the probability distribution over **X**(t) (**NOTE:** **X**(t) and **p**(t) are vectors of size n x 1). i.e. $p_1(t)$ is the probability of finding the system in state $X_1(t)$. In general, the predictive distribution for **X(t)** is quite complicated with **p**(t), being a function of all previous state variables **X**(t-1), **X**(t-2) and so on. However, if **p**(t) depends only upon the **preceding state** then the process is called **Markov Process**. A Markov process is a mathematical model that describes, in probabilistic terms, the dynamic behaviour of certain type of system over time. The change of state occurs only at the end the time period and nothing happens during the time period chosen. Thus, a Markov process is a stochastic process which has the property that the probability of a transition from a given state $p_i(t)$ to a future state $p_j(t+1)$ is dependent **only** on the present state and not on the manner in which the current state was reached. This is also called the Markovian property. If a Markov process meets the following conditions:

1. The system can be described by a set of finite states and that the system can be in one and only one state at a given time.

2 The transition probability P_{ij}, the probability of transition from state i to state j, is given from every possible combination of i and j (including i = j) and the transition probabilities are assumed to be stationary (unchanging) over the time period of interest and independent of how state i was reached. and

3. Either the initial state of the system or the probability distribution of the initial state is known.

Then, is called a **finite-state first order Markov chain.**

6.2 Finite State First Order Markov Chain Model development:

As mentioned in the introduction, to completely specify the model, all we need to know are the initial state or probability distribution of the intial state) of the system $\mathbf{p}(0) = [p_1, p_2, \dots p_n]$ and the **transition probability matrix** **P**.

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1n} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ P_{n1} & P_{n2} & P_{n3} & \cdots & P_{nn} \end{bmatrix}$$

Here, P_{ij} represent the constant probability (finite state first order Markov chain) of transition from state X_i (t) to state X_j (t+1) for any value of t. The Markovian property makes **P, time invariant.** Knowing **P**, we can also construct a transition diagram to represent the system.

Given the initial distribution **p**(0),

$$\mathbf{p}(1) = \mathbf{p}(0) \,.\, \mathbf{P}$$
$$\mathbf{p}(2) = \mathbf{p}(1).\mathbf{P} = p(0).\mathbf{P}.\mathbf{P} = p(0).\mathbf{P}^2$$

thus, for any k,

$$\mathbf{p}(k) = \mathbf{p}(0) \,.\, \mathbf{P}^k$$

We also note that the elements of **P** must satisfy the following conditions:

$$\sum_{j=1}^{j=n} P_{ij} = 1 \quad \text{for} \quad \text{all} \quad i \quad (\text{row} \quad \text{sum})$$

and $P_{ij} >= 0$ for all i and j.

AN EXAMPLE: A delicate precision instrument has a component that is subject to random failure. In fact, if the instrument is working properly at a given moment in time, then with probability 0.15, it will fail within the next 10 minute period. If the component fails, it can be replaced by a new one, an operation that also takes 10 minutes. The present supplier of replacement components does not guarantee that all replacement components are in proper working condition. The present quality standards are such that about 4% of the components supplied are defective. However, this can be discovered only after the defective component has been installed. If, defective, the instrument has to go through a new replacement operation. Assume that when failure occurs, it always occurs at the end of a 10- minute period.

a) Find the transition probability matrix associated with this process.

b) Given that it was properly working initially, what is the probability of finding the instrument not in proper working condition after 20 minutes? after 40 minutes?.

6.3 Classification of Finite Markov Chains:

Two states i and j of a system defined by the transition matrix, are said to **communicate** if j is accessible from i and i is accessible from j. The number of transitions is **not** important. Communication is a class property. If i communicates with both j and k then j communicate with k. Consider the following transition matrices:

$$\begin{array}{cccccc} & & 1 & 2 & 3 & 4 \\ & 1 & .5 & .5 & 0 & 0 \\ P_1 = & 2 & .33 & .67 & 0 & 0 \\ & 3 & .25 & .25 & .25 & .25 \\ & 4 & 0 & 0 & .25 & .75 \end{array} \qquad \begin{array}{cccccc} & & 1 & 2 & 3 & 4 \\ & 1 & 1 & 0 & 0 & 0 \\ P_2 = & 2 & 0 & 1 & 0 & 0 \\ & 3 & .5 & 0 & .2 & .3 \\ & 4 & 0 & .3 & .5 & .2 \end{array}$$

In matrix P_1 , states 1 and 2 can be reached from 3 and 4 but once in 1 or 2 it cannot return back to states 3 or 4. that is the process has been **absorbed** in the set of states 1 and 2. The set of states 1 and 2 is called an **ergodic set (closed set)** and the individual states 1 and 2 are called **ergodic states.** No matter where the process starts out, it will soon end up in the ergodic set. The states 3 and 4 are called **transient states** and the set 3 and 4 is called the **transient set**. If we are only interested in the long term behaviour of the system then we can forget about the transient states as the higher powers of the transient matrix will lead to zero in the transient states. It is possible for a process to have more than one ergodic set in a transition matrix.

In matrix P_2, the process will be absorbed in either state 1 or 2. This is evident from the one step transition probabilities p_{11} and p_{22}. A state that has this property is called an **absorbing state**. The absorbing states can be identified by the 1's in the leading diagonal of the transition matrix. Consider the following transition matrices:

$$P_3 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & ,2 & .5 & .3 \\ 2 & .1 & .2 & .7 \\ 3 & .6 & .3 & .1 \end{array} \qquad P_4 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & .7 & .3 \\ 2 & 0 & 0 & .4 & .6 \\ 3 & .2 & .8 & 0 & 0 \\ 4 & .6 & .4 & 0 & 0 \end{array}$$

In P_3, all states are communicable and form a single ergodic set. While in P_4 all the states communicate with each other and also forms an ergodic set. But, the process always moves from state 1 or 2 to state 3 or 4 and vice versa. Such a chain is called **cyclic.** The long term behaviour of such a process will remain cyclic.

A chain that is not cyclic is called **aperiodic**. A Markov chain that is aperiodic and irreducible is called **regular.**

6.4 Limiting State Probabilities (Steady State Probabilities)

It can be observed that the difference between the elements of the transition probabilities after a number of transitions tends to be zero. In other words all elements of all columns tend towards a common limit as the number of transition increases. For **regular Markov chains** *this is always the case*. But not so for cyclic chains or for chains with ergodic sets.

Theorem: If **P** is the transition matrix of a regular Markov chain, and if $\mathbf{P^k}$ approaches a unique limiting matrix, then the process is said to have reached a steady state.

Since the system reaches a steady state the kth and (k+1)th states are the same.

i.e $p(0).P^{k+1} = p(k+1) = p(k)$

thus if the steady state probabilities are $p(k) = (a_1, a_2, ... a_n)$
then $\mathbf{(a_1, a_2, ... a_n) = (a_1, a_2, ... a_n).P}$

or

$$a_1 = P_{11}.a_1 + P_{21}.a_2 + ... P_{n1}.a_n$$

$$a_2 = P_{12}.a_1 + P_{22}.a_2 + ... P_{n2}.a_n$$

$$........$$

$$a_n = P_{1n}.a_1 + P_{2n}.a_2 + ... P_{nn}.a_n$$

and

$$1 = a_1 + a_2 + ... a_n$$

(this is a necessary condition)

Discarding one of these equations we can find the values of a's.

(Note that there is one equation more than the number of variables. The first n equations are not indepndent)

The limiting state probabilities represent the approximate probabilities of finding the system in each state at the beginning (or end) of a transition after a sufficiently large number of transitions has occurred for the memory of the initial state is to be more or less lost.

Special Case 1: If **P has all rows equal** then the limiting matrix is equal to **P** itself.

Special Case 2: The transition matrix is said to be **doubly stochastic** if the row and column totals are both equal to 1 and the chain is regular. In this case the steady state probabilities of all states are equal to 1/n. Where, n = the number of states.

(Refer pp261 - 272 of Introduction to Operations Research Techniques by Daellenbach & George

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Example 1 (Contd)

c) Assume that each replacement component has a cost of $2.50, and the opportunity cost in terms of lost profits during the time instrument is not working is $15.80 per hour. What is the average cost per day?

6.5 Markov Chains with rewards.

If there is a benefit attached to each state of the process say c_i , then the long run average overall benefit is given by,

$$\text{Average Benefit} = \sum_{i=1}^{i=n} c_i . a_i$$

This can be used to determine the best policy amongst a number of alternate policies. Also used to determine the average values.

6.6: Failure Model

An example is used to illustrate the use of Markov chain modelling in failure analysis.

EXAMPLE 2: A control device contains N parallel circuit boards, all of which have to function for the device to operate properly. Each circuit board is subjected to random failure. The failure rate increases with age of the circuit boards. Past records of 122 circuit boards give the following survival function:

No. of weeks used	0	1	2	3	4	5	6	7
No. of Circuit Board Survived	122	122	116	109	98	78	39	0
% Failing (of those survived)	-	0	5	6	10	20	50	100

the following week									

Any circuit board failed is replaced by the beginning of the following week. No circuit boards survives to age 7 weeks. Thus all six week old circuit boards are replaced automatically. The failure and replacement pattern of each circuit, considered individually, over a period of time can be modelled as a Markov chain. Each week represents a transition. If there are **three** circuit boards in the device, answer the following questions:

a) What is the long term probability of failure of a single circuit board?

b) How often the **device** fail?.

c) What is the average rate of replacement of circuit boards?.

d) If each down time of the device costs \$500.00 per week, and each circuit board replaced costs \$70, what is the average weekly cost?.

e) Would a policy of forced individual replacement at an earlier age, say after 3 or 4 weeks, lower the average weekly cost?.

6.8 First Passage Time

In the analysis so far, we have only answered questions such as "what is the probability that the system will be in state i after n transitions?" or " what is the long term probability that the system will be in state i?" and so on. Frequently, we need to know "how long will it take to go from state i to state j?". Even though time is used, we can answer the question in terms of the number of transitions because the time interval is fixed. The number of transitions (time) required before the state will move from i to j for the first time is referred to as the **first passage time**. It is possible to calculate the average (or expected) number of transitions for the passage from state i to j.

Let m_{ij} be the expected first passage time (number of transitions from state i to j)

The probability of moving from i to j is P_{ij} took exactly one transition. If this is not the case then the state will change to k (<> j)

The probability of moving from i to k (for all k <> j) would be the sum of all the probabilities P_{ik} for all k (<> j). In other words,

$$\sum_{k \neq j} P_{ik}$$

We now need to move from j to j. This may require many transitions and is given by,

$$\sum_{k \neq j} m_{kj} \cdot P_{ik}$$

Thus the expected transitions from i to j is given by,

$$m_{ij} = P_{ij} + \sum_{k \neq j} P_{ik} + m_{kj} . P_{ik}$$

This reduces to

$$m_{ij} = 1 + \sum_{k \neq j} P_{ik} . m_{kj}$$

In the case when i=j, the first passage time is called '**mean recurrence time**'. The value of m_{ij} can easily be found from the steady state probabilities a_i as $m_{ij} = 1/a_i$

(Refer pp247 - 250 of Operations Research Principles & Practice by Phillips, Ravindran & Solberg)

EXAMPLE:

Consider the following Markov chain:

$$\begin{vmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.75 & 0.25 & 0 \end{vmatrix}$$

The problem is to find the expected transitions for the system to move from state 1 to 3.

Using the above equation,

$$m_{13} = 1 + P_{11} m_{13} + P_{12} m_{23}$$
$$m_{23} = 1 + P_{21} m_{13} + P_{22} m_{23}$$

Solving these two equations, we can find the required number of transitions.

The number of equations required to find the first passage time is usually (n - 1).

Back to MEN170 Contents Page