Sets and Set Operations
Sets

**Definition:** a set is an unordered collection of objects. These objects are usually referred to as *elements* or *members* of the set. (Mathematician Georg Cantor’s definition of a set.)

We use the notation $x \in S$ to indicate that $x$ is a member of $S$.

**Definition:** two sets are *equal* if and only if they have the same elements.

**Definition:** the *universal set* is denoted by $U$ and refers to the set of all objects in the currently understood domain.

**Definition:** the *empty set* is denoted by $\emptyset$ or $\{\}$ and refers to the set which contains no elements.
**Subsets**

**Definition:** a set $A$ is said to be a *subset* of set $B$ if and only if every element of $A$ is also an element of $B$. We use the notation $A \subseteq B$ to indicate “$A$ is a subset of $B$”.

**Definition (alternate):** a set $A$ is a subset of $B$ if and only if:

$$\forall x \ (x \in A) \rightarrow (x \in B)$$

**Definition:** a set $A$ is said to be a *proper subset* of $B$ if and only if $A \subseteq B$ and $A \neq B$. We denote that $A$ is a proper subset of $B$ with the notation $A \subset B$. 

Finite and Infinite Sets

**Definition:** Let $S$ be a set. If there are exactly $n$ distinct elements in $S$, where $n$ is a nonnegative integer, we say $S$ is a *finite* set, and that $n$ is the *cardinality* of $S$. The cardinality of $S$ is denoted by $|S|$.

**Definition:** a set is *infinite* if it is not finite.
Definition: Given a set S. The *power set* of S is the set of all subsets of S. The power set is denoted by P(S).
Ordered n-Tuples and Cartesian Products

**Definition:** An ordered n-tuple \((x_1, x_2, \ldots, x_n)\) is the ordered collection that has \(x_1\) as its first element, \(x_2\) as its second element, \(\ldots\), and \(x_n\) as its nth element, where \(n \geq 2\).

**Definition:** Let \(S\) and \(T\) be sets. The *Cartesian product* of \(S\) and \(T\), denoted by \(S \times T\), is the set of all ordered pairs \((x, y)\), where \(x \in S\) and \(y \in T\). In other words:

\[
S \times T = \{ (s, t) \mid s \in S \text{ and } t \in T \}.
\]
Relations

**Definition:** A subset of the Cartesian product $A \times B$ is called a *relation* from the set $A$ to the set $B$. 
Set Operations

**Definition:** Let $A$ and $B$ be sets. The *union* of $A$ and $B$, denoted by $A \cup B$, is the set that contains those elements that are either in $A$ or in $B$, or in both.

Alternate definition: $A \cup B = \{ x \mid x \in A \lor x \in B \}$

**Definition:** Let $A$ and $B$ be sets. The *intersection* of $A$ and $B$, denoted by $A \cap B$, is the set that contains those elements that are in both $A$ and $B$.

Alternate definition: $A \cap B = \{ x \mid x \in A \land x \in B \}$
Set Operations

**Definition:** Let A and B be sets. The *difference* or *complement* of A and B, denoted by \( A - B \), is the set containing those elements that are in A but not in B.

Alternate definition: \( A - B = \{ x \mid x \in A \land x \notin B \} \)

**Definition:** Let \( U \) be the universal set. Let A be a set. The *complement* of A, denoted by \( \overline{A} \), is the complement of A with respect to U. In other words, \( \overline{A} \) is \( U - A \).
Set Operations

**Definition:** Two sets are called *disjoint* if their intersection is empty.

Alternate definition: A and B are disjoint if and only if $A \cap B = \emptyset$. 
The Inclusion-Exclusion Principle: a counting technique which is a generalization of the familiar method of obtaining the number of elements in the union of two finite sets:

\[ |A \cup B| = |A| + |B| - |A \cap B| \]

For three sets:

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]

In general:

- Include the cardinalities of the single sets
- Exclude the cardinalities of the intersections of two sets
- Include the cardinalities of the intersections of three sets
- Exclude the cardinalities of the intersections of four sets

and so on.
Set Identities

Identity

\[ A \cup \emptyset = A \]

\[ A \cap U = A \]

Domination

\[ A \cup U = U \]

\[ A \cap \emptyset = \emptyset \]

Idempotence

\[ A \cup A = A \]

\[ A \cap A = A \]
Set Identities

Double Complement

- - A = A

Commutative

A ∪ B = B ∪ A
A ∩ B = B ∩ A

Associative

A ∪ (B ∪ C) = (A ∪ B) ∪ C
A ∩ (B ∩ C) = (A ∩ B) ∩ C
Set Identities

Distributive

\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]
\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]

Complement

\[ A \cup -A = U \]
\[ A \cap -A = \emptyset \]

Absorption

\[ A \cup (A \cap B) = A \]
\[ A \cap (A \cup B) = A \]
DeMorgan's Laws

- \((A \cap B) = -A \cup -B\)

- \((A \cup B) = -A \cap -B\)
Generalized Union and Intersection

**Definition:** The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

**Definition:** The intersection of a collection of sets is the set that contains those elements that are members of all sets in the collection.