Problem 1

(25 points). A wheel of fortune (like the game show) has the integers from 1 to 25 placed on it in a random manner. Show that regardless of how the numbers are positioned on the wheel, there are three adjacent numbers whose sum is at least 39. (Hint: Use a proof by contradiction. Let \(x_1, x_2, \ldots, x_{25}\) be the numbers in the order they appear on the wheel. Then \(x_1 + x_2 + x_3 < 39\). What other inequalities are there? Combine all of them to come up with a contradictory statement.)

Problem 2

(8 points each). Establish each of the following by mathematical induction:

a) \(\sum_{i=1}^{n} i(2^i) = 2 + (n - 1)2^{n+1}\)

b) \(\sum_{i=1}^{n} 2(3^{i-1}) = 3^n - 1\)

c) \(\sum_{i=1}^{n} (i)(i!) = (n + 1)! - 1\)

Problem 3

(10 points each). Let \(f : A \rightarrow A\) be an invertible function.

a) Prove the theorem \((f \circ g)^{-1} = g^{-1} \circ f^{-1}\).

b) For \(n \in \mathbb{Z}^+\) prove that \((f^n)^{-1} = (f^{-1})^n\) (This result can be used to define \(f^{-n}\) as either \((f^n)^{-1}\) or \((f^{-1})^n\).) (Hint: Use 3a and induction to solve this.)

Problem 4

(15 points). Let \(a, d\) be fixed integers. Determine a summation formula for \(a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d)\), for \(n \in \mathbb{Z}^+\). Verify your result by mathematical induction.
Problem 5

(8 points each). For \( n \in \mathbb{Z}^+ \), prove each of the following by mathematical induction: (\( x|y \) meaning that \( y \) is divisible by \( x \))

a) \( 5|(n^5 - n) \)

b) \( 6|(n^3 + 5n) \)