Mathematical Induction
Methods of Proving Theorems

Direct proof:
- $p \rightarrow q$ is proved by showing that if $p$ is True, $q$ follows.

Indirect proof:
- to prove $p \rightarrow q$, prove the contrapositive $\neg q \rightarrow \neg p$. If $\neg q$ holds, $\neg p$ follows.

Proof by contradiction:
- show that $(p \land \neg q)$ contradicts the assumptions

Proof by cases
- break the proof into cases and prove each one individually

Proof of equivalences:
- $(p \leftrightarrow q)$ is proved by proving $(p \rightarrow q) \land (q \rightarrow p)$. 
Proofs with Quantifiers

Existential Quantifiers:
1. Find an example (by making educated guesses) that shows that the statement holds.
2. Prove by contradiction: negate the existentially quantified statement and show that it implies a contradiction.
3. To disprove: choose an arbitrary $x$ and show that no matter what $x$ is, the property is False.

Universal Quantifiers:
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Mathematical Induction is used to prove statements of the form $\forall n \ P(n)$, where $n \in \mathbb{Z}^+$. 

A mathematical induction proof typically consists of two steps:

1. **Basis:** The proposition $P(1)$ is True.
2. **Inductive Step:** The implication $P(n) \rightarrow P(n+1)$ is True for all $n \in \mathbb{Z}^+$.

From this, we conclude $\forall n \ P(n)$.

How can we make this claim?
Axioms for the Set of Positive Integers

**Axiom 1.** The number 1 is a positive integer.

**Axiom 2.** If $n$ is a positive integer, then $n + 1$, called the *successor* of $n$, is also a positive integer.

**Axiom 3.** Every positive integer other than 1 is the successor of a positive integer.

**Axiom 4.** (The Well-Ordering Property) Every nonempty subset of the set of positive integers has a least element.

**Theorem.** Suppose $P(1)$ is True and $P(n) \rightarrow P(n+1)$ is True for all positive integers $n$. Then $\forall n P(n)$. 
Strong Induction

Regular induction:
• uses the Base Step $P(1)$
• uses the Inductive Step $P(n) \rightarrow P(n+1)$

Strong induction:
• uses the Base Step $P(1)$
• uses the Inductive Step $P(1)$ and $P(2)$ and … and $P(n) \rightarrow P(n+1)$