Use mathematical induction to prove all of the statements below.

**Problem 1**

(15 points) Prove that for every nonnegative integer \( n \),

\[
\sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}.
\]

**Problem 2**

(15 points) Prove that for every nonnegative integer \( n \),

\[
\sum_{i=0}^{n} i2^i = (n-1)2^{n+1} + 2.
\]

**Problem 3**

(15 points) Prove that for every positive integer \( n \), 43 divides \( 6^{n+1} + 7^{2n-1} \).

**Problem 4**

(15 points) Prove that for every positive integer \( n \), \( \frac{2^n}{n!} \leq \frac{4}{n} \).

**Problem 5**

(20 points) Suppose that a shop offers gift cards in values of 25 dollars and 30 dollars. Determine the possible total amounts you can form using these gift cards. *(Hint: use strong induction.)*

**Problem 6**

(20 points) A polygon is said to be *convex* if any line joining two vertices lies within the polygon or on its boundary. A *diagonal* is a line joining two non-adjacent vertices. Show that in a convex polygon with \( n \) vertices, the greatest number of diagonals that can be drawn is \( n(n-3)/2 \).